Birthday Problem

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Last edited: August 15, 2025

"Want to bet with me that at least two people in our two classes share the same birthday?" The math behind this is called birthday paradox (problem). There are 365 days in a year, and intuitively, it seems unlikely that any two random people would share the same birthday. But have you ever wondered, "Why do my friends on Facebook so often share the same birthday?" Is it just a coincidence?

Definitely not! In this essay, we dive into this phenomenon and give a mathematical explanation to understand just how likely it really is. Let's first make our main question precise.

Problem 1. Given any group of n people, what is the probability that at least two of them share the same birthday?

Solution. In finding the probability that *at least two* people share a birthday, a natural strategy is to first calculate the probability that no one shares a birthday, and then subtract that from 1. So let's first compute the number of ways that all n people have different birthdays.

- The first person has no restrictions, so there are 365 possibilities.
- The second person cannot have the same birthday as the first one, so there are 364 possibilities.
- The third person cannot have the same birthday as the previous two, so there are 363 possibilities.
- ...
- The n-th person cannot have the same birthday as the previous $n-1$ people, so there are $365-(n-1)=365-n+1$ possibilities.

Thus, the total number of all favorable cases is

$$365 \times 364 \times 363 \times \cdots \times (365-n+1).$$

Meanwhile, the total number of all possible birthday assignments is

$$365^n.$$

Therefore, the probability that no one shares a birthday is

$$\frac{365 \times 364 \times 363 \times \cdots \times (365 - n + 1)}{365^n}.$$

Finally, the probability that at least two people share a birthday is

$$P(n) = 1 - \frac{365 \times 364 \times 363 \times \cdots \times (365 - n + 1)}{365^n}.$$

□

Plugging in different values of n, we obtain the following table.

n	$P(n)$
10	11.69%
20	41.14%
23	50.73%
50	97.04%
57	99.01%
60	99.41%

What does this tell us?

- With only 10 people, the probability that at least two share a birthday is just 11.69%.
- But with 20 people, it already jumps to 41.14%.
- More surprisingly, with just 23 people, the chance exceeds 50%.
- With 57 people, the chance that at least two share a birthday is over 99%!

So when someone says, "Want to bet 1,000 bucks that someone in our two classes shares the same birthday?"—you know you shouldn't take the bet, because the odds are overwhelmingly high. Of course, 99% is not a guarantee. If you want to be 100% sure that a birthday is shared, then by the pigeonhole principle, you need 366 people. But if we're talking about bets, 99% is pretty convincing, right?

Through this example, I want to tell you that many things we think of as "coincidences" might not actually be all that surprising. There's often a mathematical explanation behind them, and what makes them so extraordinary is that we tend to look at the problem from the wrong perspective. When something "coincidental" happens to someone, we mistakenly think in terms of the probability that it happens to *that particular person*. What we really should be asking is, "What are the odds that it happens to *someone*?"

Let's revisit the birthday example. We were considering the probability that *any two people* share the same birthday—not that *a specific person*, like you, shares a birthday with someone else. These are very different questions. We have seen that the former has a high probability. But if we ask instead, "What's the chance that *you* share a birthday with someone else?"—that's much lower. For this probability to exceed 50%, you would need 253 people. (Exercise!)

Let me tell a story to illustrate this idea.[1] Joan Ginther, an American woman, won the lottery four times in her lifetime:

- 5.4 million in 1993,
- 2 million in 2006,
- 3 million in 2008, and
- 10 million in 2010.

This is a total of 20.4 million in winnings.

You might think, "How outrageous is it to win million-dollar lotteries four times?" But again, just like what I said earlier, the question "What's the chance that *she* (or I) wins the lottery four times?" is very different from "What's the chance that *someone in the world* wins the lottery four times?" The former is unimaginably small, but the latter? Consider how many people in the world history play the lottery, how many different lotteries there are, how frequently they're drawn, how long they've existed, and the fact that someone who wins once is more likely to keep playing. With all that in mind, ask yourself again, "What's the chance that someone, somewhere in human history, wins the lottery four times?" Suddenly, it doesn't seem so miraculous anymore, does it?

[1] Listen to this podcast episode Hidden Brain – What Are The Odds.