Christmas Stocking Theorem

Timo Chang
timo65537@protonmail.com

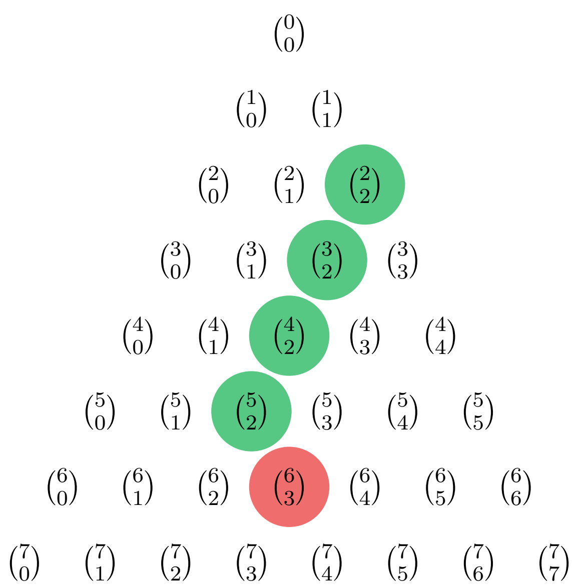
Last edited: August 15, 2025

In this essay, we learn an interesting theorem in combinatorics.

Theorem 1 (Christmas Stocking Theorem)**.** *For $n, k \in \mathbb{N}$, $n \geq k$, we have*

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$

So where is the Christmas stocking? If we consider the Pascal's triangle in the notations of binomial coefficients, and color the corresponding entries from Theorem 1, then you will immediately see it. The figure below shows the example of $n = 5$ and $k = 2$.



Theorem 1 then says that the green parts sum to the red part. How nice it is!

We give two proofs of Theorem 1, beginning with a combinatorial double counting argument.

First Proof of Theorem 1. Note that the right-hand side of Theorem 1 represents the number of ways to choose $k+1$ people from $n+1$ people. So to prove the theorem, we want to explain the left-hand side in the same way.

Let us consider the following scenario. We label these $n+1$ people from 1 to $n+1$. Starting with the first person, then the second, and so on, we decide whether to choose each person. And we continue making decisions until we have chosen $k+1$ people in total. Then by considering the number of the last chosen person, there are multiple possibilities:

- The $(k+1)$-st person is the last chosen one.
- The $(k+2)$-nd person is the last chosen one.
- $\ldots$
- The $(n+1)$-st person is the last chosen one.

We note that the first case indeed starts with the $(k+1)$-st person because we want to choose exactly this many people.

Now, consider the first case. If the $(k+1)$-st person is the last chosen one, then before choosing him, we must choose k people from those before him, and there are k such people. Thus, the number of ways (to choose $k+1$ people from $n+1$ people) in this case is

$$\binom{k}{k}.$$

Next, consider the second case. Similarly, if the $(k+2)$-nd person is the last chosen one, then before choosing him, we must choose k people from those before him, and there are $k+1$ such people. Thus, the number of ways in this case is

$$\binom{k+1}{k}.$$

Proceeding in this way, we see that in the last case, the number of ways is

$$\binom{n}{k}.$$

Summing up, we see that the number of ways to choose $k+1$ people from $n+1$ people is also equal to

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k}.$$

This completes the proof. □

Our next proof is algebraic and relies on the following lemma.

Lemma 2 (Pascal's Rule)**.** *For $n, r \in \mathbb{N}$, $n \geq r$, we have*

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}.$$

Proof. By the definition of binomial coefficients, we see that

$$\begin{aligned}
\binom{n}{r-1} + \binom{n}{r} &= \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!} \\
&= \frac{r \cdot n!}{r!(n-r+1)!} + \frac{(n-r+1) \cdot n!}{r!(n-r+1)!} \\
&= \frac{(n+1) \cdot n!}{r!(n-r+1)!} \\
&= \frac{(n+1)!}{r!(n+1-r)!} \\
&= \binom{n+1}{r}.
\end{aligned}$$

□

Second Proof of Theorem 1. We note that

$$\binom{k}{k} = 1 = \binom{k+1}{k+1}.$$

By applying Lemma 2 repeatedly, we have

$$\begin{aligned}
\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} &= \binom{k+1}{k+1} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} \\
&= \binom{k+2}{k+1} + \binom{k+2}{k} + \cdots + \binom{n}{k} \\
&= \cdots \\
&= \binom{n}{k+1} + \binom{n}{k} \\
&= \binom{n+1}{k+1}.
\end{aligned}$$

□