A Short and Simple Proof of Pythagorean Theorem

Timo Chang
timo65537@protonmail.com

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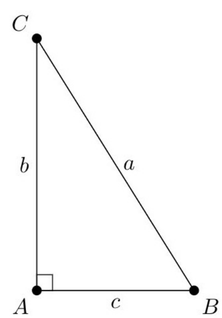
Pythagorean theorem, probably the most well-known theorem among math amateurs, has already been discovered over 300 proofs. In this essay, we demonstrate the one given by James Garfield, who was the 20th president of the United States. It is extremely simple and yet so elegant that any elementary school student can understand.

Let us first state the theorem for completeness.

Theorem 1 (Pythagorean Theorem)**.** *Consider the following right triangle.*



The hypotenuse has length a, and the two legs have length b and c. Then we have

$$a^2 = b^2 + c^2.$$

Proof. The idea of Garfield's marvelous proof is to find the area of a certain trapezoid in two different ways and create a useful equality. We duplicate our right triangle, rotate it by 90 degrees clockwise, and paste them as in Figure 1.

Let's take a closer look at what's inside.

- The original triangle is called $\triangle ABC$. And the new one is called $\triangle A'B'C'$. Note that the points B' and C are attached.
- Since these two triangles are congruent, they must share the same area. Let's call it $\mathcal{A}$.

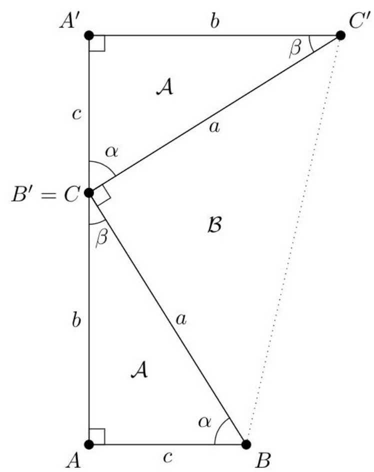


Figure 1

- Moreover, note that $\angle ABC = \angle A'B'C'$. We denote it as α. And $\angle ACB = \angle A'C'B'$. We denote it as β. Since the interior angles of a triangle add up to $180°$, we know $\alpha + \beta = 90°$.
- We draw a dashed line connecting B and C', and form a new triangle $\triangle BCC'$. Its area is denoted by $\mathcal{B}$.
- Since $\alpha + \beta = 90°$, we have $\angle BCC' = 90°$. Hence, $\triangle BCC'$ is a right triangle. And both legs have length a.

We have obtained a trapezoid $ABC'A'$. Let us compute its area in two different ways. First, using the formula of trapezoids, we know its area is

$$\frac{(b+c)\cdot(b+c)}{2}. \tag{1}$$

On the other hand, its area is also equal to $\mathcal{A} + \mathcal{A} + \mathcal{B}$. Using the formula of triangles, we know they are

$$\frac{bc}{2} + \frac{bc}{2} + \frac{a^2}{2}. \tag{2}$$

Since (1) and (2) are the same, we have

$$\frac{(b+c)\cdot(b+c)}{2} = \frac{bc}{2} + \frac{bc}{2} + \frac{a^2}{2}.$$

Clearing the denominators out and expanding the left-hand side give us

$$b^2 + 2bc + c^2 = 2bc + a^2.$$

Hence,

$$b^2 + c^2 = a^2.$$

This completes the proof. □