Visual Proof: Sum of First n Squares

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Every high school student should be familiar with this famous summation formula.

Proposition 1. *For any $n \in \mathbb{N}$, we have*

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Maybe you still remember how to "verify" this equation using mathematical induction, but none of your teacher has told you how to derive this formula from scratch. In this essay, we will see a simple explanation of it by using only manipulation of shapes.

Proof. We take $n = 5$ as an example. The general case will follow similarly. Note that finding $1^2 + 2^2 + 3^2 + 4^2 + 5^2$ is exactly the same as finding the total area of five squares with sides 1 to 5. Motivated by this, we draw three copies of them with one colored in advance as follows.

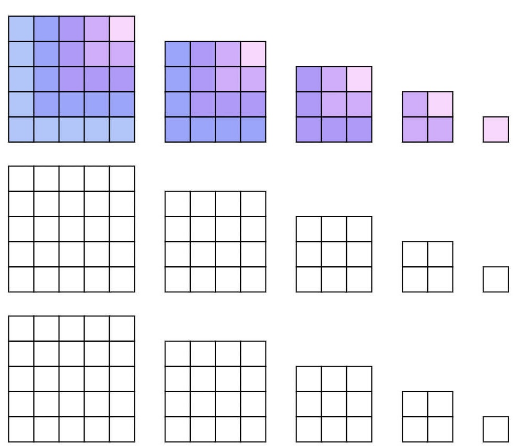

Figure 1

Next, we cut these squares and put them together, as shown in Figure 2. Pieces with colors are assembled into a tower. White squares with bold sides are those without any coloring.

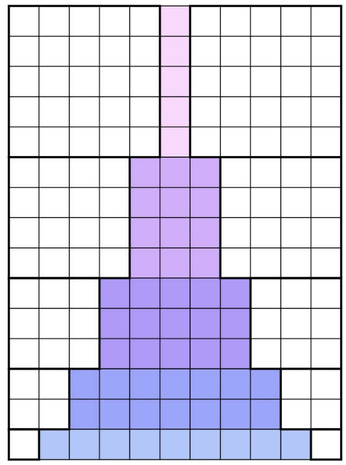

Figure 2

Since we are just cutting and recomposing, the area of this big rectangle must remain the same as those 15 squares in Figure 1, which is

$$3 \cdot (1^2 + 2^2 + 3^2 + 4^2 + 5^2). \tag{1}$$

On the other hand, let us look at Figure 2 very carefully. Its vertical side has length $1+2+3+4+5$. And the horizontal side has length equal to the number of pieces of small light blue squares adding 2. If we observe the largest square with colors in Figure 1, we see that there are $5+5-1$ small light blue squares. So the length of the horizontal side is $(2 \cdot 5 - 1) + 2 = 2 \cdot 5 + 1$. Thus, the area of the big rectangle in Figure 2 is equal to

$$(1+2+3+4+5) \cdot (2 \cdot 5 + 1) = \frac{5(5+1)}{2} \cdot (2 \cdot 5 + 1). \tag{2}$$

(Here, I have used the formula of sum of n consecutive integers.)

Since both (1) and (2) represent the area in Figure 2, they must be equal. That is,

$$3 \cdot (1^2 + 2^2 + 3^2 + 4^2 + 5^2) = \frac{5(5+1)}{2} \cdot (2 \cdot 5 + 1).$$

Dividing both sides by 3 gives us

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \frac{5(5+1)(2 \cdot 5 + 1)}{6}.$$

And here it is, our desired formula. $\square$