Liouville's Theorem on Diophantine Approximation and Liouville Number

Timo Chang
timo65537@protonmail.com

Last edited: November 6, 2024

The concepts of rationality and transcendence of numbers had already been held by mathematicians centuries ago. Some progress concerning the rationality was made, for example, by Euler in 1744 that e is irrational, and by Lambert in 1761 that so is the number π. However, the existence of transcendental numbers was not known until 1844 when Joseph Liouville made the first breakthrough in a certain approximation property of irrational algebraic numbers. Using that, he was able to construct the very first example of transcendental numbers explicitly. In this essay, we aim to present his initiation of the whole theory.

1 Liouville's Theorem on Diophantine Approximation

Theorem 1.1 (Liouville's Theorem on Diophantine Approximation)**.** *Let α be an irrational algebraic number. Suppose α satisfies an irreducible polynomial $f(x) = \sum_{i=0}^{n} c_i x^i \in \mathbb{Z}[x]$ with degree n. Then there exists an $A > 0$ such that for any $p, q \in \mathbb{Z}$ and $q > 0$, we have*

$$\left|\alpha - \frac{p}{q}\right| > \frac{A}{q^n}.$$

Roughly speaking, the theorem says that every irrational algebraic number α can not be approximated very "nicely" by rational numbers, because there is always a gap A/q^n between α and any rational number p/q. In other words, if an irrational number α can be approximated nicely (to certain degree) by rational numbers, then it must be transcendental.

Proof. Since f is a polynomial, clearly we have f' is a continuous function. So by extreme value theorem, $M := \max_{[\alpha-1,\alpha+1]} |f'(x)|$ is finite. Let $\{\alpha_1, \ldots, \alpha_m\}$ be the set of distinct roots of f other than α, and set

$$0 < A < \min\{1, 1/M, |\alpha - \alpha_1|, \ldots, |\alpha - \alpha_m|\}.$$

We claim that this A satisfies our desired property.

Suppose on the contrary, there exist $p, q \in \mathbb{Z}$ and $q > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| \le \frac{A}{q^n}. \tag{1}$$

It follows that

$$\left|\alpha - \frac{p}{q}\right| \le \frac{A}{q^n} \le A < 1,$$

and so $p/q \in [\alpha - 1, \alpha + 1]$. By mean value theorem, we have

$$f(\alpha) - f\left(\frac{p}{q}\right) = f'(\xi)\left(\alpha - \frac{p}{q}\right)$$

for some ξ between α and p/q. Note that $\xi \in [\alpha - 1, \alpha + 1]$ because p/q also lies in this interval. Consequently, we have

$$M \ge |f'(\xi)|$$

by our definition of M.

Note that $f(\alpha) = 0$ because α is a root of f. And from $|\alpha - p/q| \le A < |\alpha - \alpha_i|$ for all $i = 1, \ldots, m$ we know $p/q \notin \{\alpha_1, \ldots, \alpha_m\}$. In other words, $f(p/q) \neq 0$. These two imply that $f'(\xi) \neq 0$. So we may write

$$\left|\alpha - \frac{p}{q}\right| = \frac{|f(\alpha) - f(p/q)|}{|f'(\xi)|} = \frac{|f(p/q)|}{|f'(\xi)|}.$$

We see that the numerator

$$\begin{aligned} |f(p/q)| &= \left|c_0 + c_1\frac{p}{q} + \cdots + c_n\left(\frac{p}{q}\right)^n\right| \\ &= \frac{1}{q^n}\left|c_0 q^n + c_1 p q^{n-1} + \cdots + c_n p^n\right| \\ &\ge \frac{1}{q^n}. \end{aligned}$$

The last equality is because $c_0, \ldots, c_n, p, q$ are all integers and their combination in the last absolute value is non-zero (as $f(p/q) \neq 0$). We can now combine all inequalities together and see that

$$\left|\alpha - \frac{p}{q}\right| = \frac{|f(p/q)|}{|f'(\xi)|} \ge \frac{1}{|f'(\xi)|q^n} \ge \frac{1}{Mq^n} > \frac{A}{q^n}.$$

This contradicts to (1). □

2 Liouville Number

In the last section we proved an important theorem describing an approximation property of irrational algebraic numbers due to Liouville. This enabled him to construct the very first example of transcendental numbers. Now, we're in position to see how this was done.

Definition 2.1 (Liouville Number)**.** A number $\alpha \in \mathbb{R}$ is called a *Liouville number* if for any $n \in \mathbb{N}$, there exist $a, b \in \mathbb{Z}$ and $b > 1$ such that

$$0 < \left|\alpha - \frac{a}{b}\right| < \frac{1}{b^n}. \tag{2}$$

By definition, every Liouville number is approximated nicely by rational numbers to some extent. In view of Theorem 1.1, this suggests that every Liouville number is transcendental. And this is indeed the case. We will prove this in two steps.

Proposition 2.2. *All Liouville numbers are irrational.*

Proof. We suppose on the contrary that there exists a Liouville number α which is rational. Then write $\alpha = p/q$ where $p, q \in \mathbb{Z}$ and $q \geq 1$. Take $n \in \mathbb{N}$ large enough so that $2^{n-1} > q$. We claim that this particular n contradicts to Definition 2.1. That is, we claim that for any $a, b \in \mathbb{Z}$ and $b > 1$, one of the inequalities in (2) fails. (So that α would not be a Liouville number.)

Note that $|\alpha - a/b| = |p/q - a/b| = |(pb - aq)/qb|$.

- Case 1: $|pb - aq| = 0$. Then $|\alpha - a/b| = 0$. This contradicts to the first inequality in (2).
- Case 2: $|pb - aq| \geq 1$. Then

  $$\left|\alpha - \frac{a}{b}\right| \geq \frac{1}{bq} > \frac{1}{b2^{n-1}} \geq \frac{1}{b \cdot b^{n-1}} = \frac{1}{b^n}.$$

  The second inequality is due to our choice of n, and the third inequality is because $b \geq 2$. But then this contradicts to the second inequality in (2).

This completes the proof. □

Proposition 2.3. *All Liouville numbers are transcendental.*

Proof. We again suppose on the contrary that there exists a Liouville number α which is algebraic. By Proposition 2.2 we know α is irrational. So it satisfies the assumption of Theorem 1.1. Hence there exists an $A > 0$ such that for any $p, q \in \mathbb{Z}$ and $q > 0$, we have

$$\left|\alpha - \frac{p}{q}\right| > \frac{A}{q^n} \tag{3}$$

for some $n \in \mathbb{N}$.

We take $r \in \mathbb{N}$ large enough so that $2^r > 1/A$. Then since α is a Liouville number, by Definition 2.1, we can find $a, b \in \mathbb{Z}$ and $b > 1$ such that $0 < |\alpha - a/b| < 1/b^{n+r}$. But this implies

$$0 < \left|\alpha - \frac{a}{b}\right| < \frac{1}{b^{n+r}} = \frac{1}{b^n b^r} \leq \frac{1}{b^n 2^r} < \frac{A}{b^n},$$

which contradicts to (3). This completes the proof. □

We have shown that all Liouville numbers are transcendental. An example of such numbers was given by Liouville himself, called *Liouville's constant.*

Example 2.4 (Liouville's Constant)**.** Define the infinite summation

$$L := \sum_{m=1}^{\infty} \frac{1}{10^{m!}} = 0.11000100000\cdots .$$

Observe that its decimal representation has 1 at every $m!$-th digit and 0 otherwise. In particular, L is irrational. (Recall that a real number is rational if and only if its decimal representation is either repeating or terminating.) We claim that L is a Liouville number.

Given any $n \in \mathbb{N}$, write

$$L = \sum_{m=1}^{n} \frac{1}{10^{m!}} + \sum_{m=n+1}^{\infty} \frac{1}{10^{m!}} = \frac{a'}{10^{n!}} + \sum_{m=n+1}^{\infty} \frac{1}{10^{m!}}$$

where a' is whatever it takes to make the first sum become a single fraction. Now we take $a := a'$ and $b := 10^{n!}$. It follows that

$$\begin{aligned} 0 < \left| L - \frac{a}{b} \right| &= \sum_{m=n+1}^{\infty} \frac{1}{10^{m!}} = \frac{1}{10^{(n+1)!}} + \frac{1}{10^{(n+1)!}} \sum_{m=n+2}^{\infty} \frac{1}{10^{m!-(n+1)!}} \\ &\leq \frac{1}{10^{(n+1)!}} + \frac{1}{10^{(n+1)!}} \sum_{m=1}^{\infty} \frac{1}{2^m} = \frac{2}{10^{(n+1)!}} < \frac{1}{10^{(n+1)!-1}} \leq \left(\frac{1}{10^{n!}} \right)^n \end{aligned}$$

where the last inequality is due to the fact that $(n+1)! - 1 \geq n(n!)$ for all $n \in \mathbb{N}$. So by Definition 2.1, L is a Liouville number. We may now conclude that L is transcendental by Proposition 2.3.