Problem Set

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Last edited: November 29, 2024

- Rings are assumed to have identity, but may not be commutative unless otherwise specified.
- Ring homomorphisms always send 1 to 1.

1 Problems

Problem 1. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules (A being commutative). Suppose M'' is flat. Show that the induced sequence $0 \to M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0$ is still exact for any A-module N.

Problem 2. Let A be a commutative ring.

(a) Let F be a free A-module of rank $n \geq 2$ with basis $\{e_1, \ldots, e_n\}$. Show that $e_i \otimes e_j + e_j \otimes e_i$ is not a pure tensor in $F \otimes_A F$ for all $i \neq j$.

(b) Find an element in $\mathrm{Mat}_n(A)$ which is not of the form vw^t where v, w are viewed as column vectors.

Problem 3. Let G be a group with normal subgroups H, K.

(a) Find examples to illustrate that $H \simeq K$ may not imply $G/H \simeq G/K$ and vice versa.

(b) Suppose there exists $\phi \in \mathrm{Aut}(G)$ such that $\phi(H) = K$. Show that it induces an isomorphism $\overline{\phi} : G/H \to G/K$.

Problem 4. Suppose $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ are normal subgroups such that $H_1 \simeq H_2$ and $G_1/H_1 \simeq G_2/H_2$. Does there always exist $\phi \in \mathrm{Iso}(G_1, G_2)$ such that the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H_1 & \longrightarrow & G_1 & \longrightarrow & G_1/H_1 & \longrightarrow & 1 \\ & & \downarrow \wr & & \downarrow \phi & & \downarrow \wr & & \\ 1 & \longrightarrow & H_2 & \longrightarrow & G_2 & \longrightarrow & G_2/H_2 & \longrightarrow & 1 \end{array}$$

commutes? (The rows are natural short exact sequences.) Do the same thing for R-modules where R is a ring.

Problem 5. Show that S_5 has no subgroup of order 30.

Problem 6. Given a polynomial $f(x)$ over a field F. Suppose a is not a root of $f(x)$. Let $N \in \mathrm{Mat}_n(F)$ be a nilpotent matrix. Show that the matrix polynomial $f(aI_n + N)$ is invertible.

Problem 7. Let R be a PID. Say $M = R/(d_1) \oplus \cdots \oplus R/(d_m)$ with $d_1 \mid \cdots \mid d_m$ are non-zero and non-unit. Is it possible that $M \simeq R/(c_1) \oplus \cdots \oplus R/(c_n)$ with $n < m$? (The familiar condition $c_1 \mid \cdots \mid c_n$ is not required.)

Problem 8 (Finite Topology). Let X, Y be sets. We identify the set of functions from X to Y with the product set Y^X by $f \mapsto (f(x))_{x \in X}$, and endow it with the product topology where each Y is given the discrete topology. We call this topology the *finite topology*.

(a) Show that a base of open sets of Y^X consists of

$$\mathcal{O}_{f,\{x_1,\dots,x_n\}} := \{g : X \to Y \mid g(x_i) = f(x_i), i = 1, \dots, n\}.$$

for some $x_1, \dots, x_n \in X$. ($\mathcal{O}_{f,\{x_1,\dots,x_n\}}$ is an open neighborhood of f.)

(b) Suppose further that X, Y are groups. Show that the subset $\mathrm{Hom}(X, Y)$ of group homomorphisms from X to Y is closed in Y^X.

(c) Let E/F be an algebraic extension. Show that $\mathrm{Aut}(E/F)$ is closed in E^E.

Problem 9. Let M, N be two modules over a commutative ring A. Show that the following are equivalent.

(i) $M = N$.

(ii) $M_{\mathfrak{p}} = N_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$.

(iii) $M_{\mathfrak{m}} = N_{\mathfrak{m}}$ for all $\mathfrak{m} \in \mathrm{Max}(A)$.

In other words, being "equal" is a local property. (Be careful that "=" cannot be replaced by "$\simeq$". For example, consider any two non-isomorphic *invertible* A-modules (see [Eis95, Section 11.3]). Why this doesn't contradict to [Ati69, Proposition 3.9]?)

Problem 10 (Poincaré Theorem). Let G be a group. Suppose H_1, H_2 are two subgroups such that $[G : H_1]$ and $[G : H_2]$ are both finite. Show that $[G : H_1 \cap H_2]$ is also finite.

Problem 11. Let E/F be a finite extension. For each $\alpha \in E$, consider the F-linear operator $T_\alpha : E \to E$ given by $T_\alpha(v) := \alpha \cdot v$ for every $v \in E$. Show that the minimal polynomial $p(x)$ of T_α is equal to the minimal polynomial $\mathrm{Irr}_F(\alpha)$ of α over F, and the characteristic polynomial $f(x)$ of T_α is equal to $\mathrm{Irr}_F(\alpha)^{[E:F(\alpha)]}$. (Use either linear algebra or module theory.)

Problem 12 (Perfect Pairing). Let R be a commutative ring and M, N be two R-modules. An R-bilinear map (sometimes called a *pairing*) $B : M \times N \to R$ is said to be *perfect on the left* if the induced R-linear map $M \to \mathrm{Hom}_R(N, R)$ is an isomorphism. In other words, M is isomorphic to the dual module of N. Perfect on the right is defined in a similar way. We call B a *perfect pairing* if it is perfect on both left and right. (Note some authors use the term "perfect" even when B is only one-sided perfect.)

(a) Find an example to show that this definition is asymmetric, i.e., perfect on the left is not equivalent to perfect on the right.

(b) Find an example to show that non-degeneracy does not necessarily imply perfection.

Problem 13 (Dual Basis). Let V, W be two finite dimensional F-vector spaces with a bilinear form $B : V \times W \to F$. Show that the following are equivalent.

(i) B is non-degenerate.
(ii) B is perfect (Problem 12).
(iii) For any basis $\beta = \{v_1, \ldots, v_n\}$ of V, there exists a unique basis $\gamma = \{w_1, \ldots, w_n\}$ of W such that $B(v_i, w_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. (Such (β, γ) is called a dual basis of B.)
(iv) The matrix $[B(v_i, w_j)]$ is invertible for some basis $\{v_1, \ldots, v_n\}, \{w_1, \ldots, w_n\}$ of V, W, respectively.

Problem 14. Find an element which is neither separable nor purely inseparable over a field F.

Problem 15. Let p, q be two prime numbers. Let $f(x) = x^p - q \in \mathbb{Q}[x]$ and K be the splitting field of f over $\mathbb{Q}$. Find $\mathrm{Gal}(K/\mathbb{Q})$.

Problem 16 (Perfect Field). Let F be a field of characteristic $p > 0$. Show that the following are equivalent.

(i) Every algebraic extension of F is separable.
(ii) $F = F^p := \{a^p \mid a \in F\}$.
(iii) The Frobenius endomorphism $\mathrm{Frob}_p : x \mapsto x^p$ is an automorphism.

We call F a *perfect field* if one of the above conditions is satisfied.

Problem 17. Find an example of a field extension E/F such that $E \simeq F$ but $[E : F] > 1$.

Problem 18. Let K, L be two number fields. Suppose there exists a prime number p such that p is unramified in K and totally ramified in L. Show that K and L are linearly disjoint over $\mathbb{Q}$.

Problem 19. Let $k := \mathbb{F}_q(t)$ be the rational functional field over a finite field $\mathbb{F}_q$ and $k_\infty := \mathbb{F}_q((1/t))$ be the field of formal Laurent series. Show that k_∞/k is separable. (This is actually true in a more general setting.)

Problem 20 (Triangular Ring). Let R, S be two rings and M be an (R, S)-bimodule. Set

$$A := \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \,\middle|\, r \in R, m \in M, s \in S \right\}.$$

We identify A as the additive group $R \oplus M \oplus S$, and give the latter a ring structure according to the multiplication on A. Show that

(a) The left ideals of A are of the form $I_1 \oplus I_2$, where I_2 is a left ideal in S, and I_1 is a left R-submodule of $R \oplus M$ containing MI_2.

(b) The right ideals of A are of the form $J_1 \oplus J_2$, where J_1 is a right ideal in R, and J_2 is a right S-submodule of $M \oplus S$ containing $J_1 M$.

(c) The (two-sided) ideals of A are of the form $K_1 \oplus K_0 \oplus K_2$, where K_1 is an ideal in R, K_2 is an ideal in S, and K_0 is an (R, S)-subbimodule of M containing $K_1 M + MK_2$.

Problem 21. Find a ring which is right artinian but not left artinian.

Problem 22 (Quaternion Algebra). Let F be a field of $\mathrm{char}(F) \neq 2$. For $a, b \in F^\times$, define the *quaternion algebra* $Q(a,b) := F \oplus F\mathbf{i} \oplus F\mathbf{j} \oplus F\mathbf{k}$, where the multiplication law is F-linearly spanned by

$$\mathbf{i}^2 = a, \quad \mathbf{j}^2 = b, \quad \mathbf{k} = \mathbf{ij} = -\mathbf{ji}.$$

Show that

(a) $Q(a,b)$ is a (4-dimensional) central simple algebra over F. In particular, $Q(a,b)$ is either isomorphic to a central division algebra of dimension 4 or $\mathrm{Mat}_2(F)$.

(b) Conversely, every 4-dimensional central simple algebra over F is isomorphic to $Q(a,b)$ for some $a, b \in F^\times$.

(c) $Q(a,b) \simeq \mathrm{Mat}_2(F)$ if and only if the quadratic form $z^2 = ax^2 + by^2$ has non-zero solutions $(x,y,z) \in F^3$. (See the definition of *Hilbert symbol* in [Ser73, Chapter III] or [Mil, Chapter III.4].)

(d) $Q(a,b) \simeq Q(b,a)$. What does it tell us in terms of (c)?

(e) $Q(a,b) \otimes_F Q(a,c) \simeq Q(a,bc) \otimes_F Q(c, -a^2 c) \simeq Q(a,bc) \otimes_F \mathrm{Mat}_2(F)$. In particular, this together with (d) imply that

$$[Q(a,b)] * [Q(a,b')] = [Q(a,bb')] \quad \text{and} \quad [Q(a,b)] * [Q(a',b)] = [Q(aa',b)]$$

in the Brauer group $\mathrm{Br}(F)$ of F. In other words, the map $Q : F^\times \times F^\times \to \mathrm{Br}(F)$ is bimultiplicative.

(f) $Q(a,b) \simeq Q(a,b)^{\mathrm{op}}$. In particular, $[Q(a,b)]$ is 2-torsion in $\mathrm{Br}(F)$.

(See [FD93, Exercise 4.15-4.27] and [Mil, Exercise IV.5.7] also.)

Problem 23. Recall Maschke's theorem: Let G be a finite group and F be a field with $\mathrm{char}(F) \nmid \#(G)$. Then every representation of G over F is completely reducible (i.e., it is a direct sum of irreducible subrepresentations). Find examples to illustrate that both conditions "G is finite" and "$\mathrm{char}(F) \nmid \#(G)$" cannot be dropped.

Problem 24. Determine the character tables of the dihedral group D_4 and the quaternion group Q_8. Conclude that character table does not determine the group uniquely (up to isomorphism).

Problem 25. Let G be a finite group with an abelian subgroup H. Show that every irreducible representation of G over $\mathbb{C}$ has degree $\le [G : H]$.

Problem 26. Let G be a finite group and (ρ, V) be an irreducible representation of G over $\mathbb{C}$ with character χ_ρ. Let $C(G)$ be the set of conjugacy classes of G.

(a) Show that for each $C \in C(G)$, the complex number

$$\sum_{g \in C} \frac{\chi_\rho(g)}{\deg \rho}$$

is integral over $\mathbb{Z}$.

(b) Show that

$$\frac{\#(G)}{\deg \rho} = \sum_{C \in C(G)} \left(\sum_{g \in C} \frac{\chi_\rho(g)}{\deg \rho} \cdot \overline{\chi_\rho(g)} \right).$$

Conclude that $\deg \rho \mid \#(G)$.

Problem 27 (Semi-direct Product). (a) Let H and N be two groups. Given a homomorphism $\varphi : H \to \mathrm{Aut}(N)$. We define the *semi-direct product* $N \rtimes_\varphi H$ to be the set $N \times H$ with the multiplication law

$$(n_1, h_1) \cdot (n_2, h_2) := (n_1 \varphi_{h_1}(n_2), h_1 h_2)$$

for all $n_1, n_2 \in N$ and $h_1, h_2 \in H$. Prove that $N \rtimes_\varphi H$ is a group under this multiplication.

(b) Let G be a group and H, N be subgroups with N normal. Assume $H \cap N = \{e\}$. Define $\varphi : H \to \mathrm{Aut}(N)$ by $\varphi_h(n) := hnh^{-1}$. Prove that $NH \simeq N \rtimes_\varphi H$.

Problem 28 (Splitting Lemma for Groups). A short exact sequence

$$1 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 1$$

of groups is said to be *left split* (resp. *right split*) if there exists a homomorphism $\alpha' : G \to H$ such that $\alpha' \circ \alpha = \mathrm{id}_H$ (resp. $\beta' : K \to G$ such that $\beta \circ \beta' = \mathrm{id}_K$).

(a) Show that the exact sequence is left split if and only if there is an isomorphism $f : G \to H \times K$ such that the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & H & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & K & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
1 & \longrightarrow & H & \xrightarrow{\iota} & H \times K & \xrightarrow{\pi} & K & \longrightarrow & 1
\end{array}$$

commutes, where the bottom row is the natural short exact sequence for direct product.

(b) Show that the exact sequence is right split if and only if there is a homomorphism $\varphi : K \to \mathrm{Aut}(H)$ and an isomorphism $f : G \to H \rtimes_\varphi K$ (Problem 27) such that the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & K & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & H & \xrightarrow{\iota} & H \rtimes_\varphi K & \xrightarrow{\pi} & K & \longrightarrow & 1 \end{array}$$

commutes, where the bottom row is the natural short exact sequence for semi-direct product.

(c) In particular, splitting on the left implies splitting on the right. Find an example to illustrate that the converse may not hold. (Recall that when G is abelian, i.e., a $\mathbb{Z}$-module, then these two conditions are equivalent.)

Problem 29. (a) Let

$$1 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 1$$

be a short exact sequence of finite groups. Suppose H is of order m, K is cyclic of order n, and $\gcd(m, n) = 1$. Show that the exact sequence is right split (Problem 28).

(b) For an odd prime number p, show that $\prod_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} a = -1$ for all $n \in \mathbb{N}$.

(c) For an irreducible polynomial $P \in \mathbb{F}_q[T]$ where q is a power of some odd prime number, show that $\prod_{a \in (A/P^nA)^\times} a = -1$ for all $n \in \mathbb{N}$.

Problem 30. Let F be a field and $f(x), g(x) \in F[x]$ which are relatively prime polynomials. Show that the field extension $F(x)$ over $F(f(x)/g(x))$ of function fields is finite and find the degree.

Problem 31. (a) Let V be a finite-dimensional vector space over F with a perfect pairing $B : V \times V \to F$. Let $f, g \in \mathrm{End}_F(V)$ and $\beta = \{v_1, \ldots, v_n\}, \gamma = \{w_1, \ldots, w_n\}$ be any dual basis of B (Problem 13). Show that the sum

$$\sum_{i=1}^{n} f(v_i)g(w_i)$$

is independent of the choice of dual basis (β, γ) of B.

(b) Suppose further that there exists $T \in \mathrm{End}_F(V)$ satisfying $B(T(v), w) = B(v, T(w))$ for all $v, w \in V$. Show that

$$\sum_{i=1}^{n} f(T(v_i))g(w_i) = \sum_{i=1}^{n} f(v_i)g(T(w_i)).$$

References

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