Riemann's functional equation of Riemann zeta function

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Recall the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1. \tag{1}$$

In this essay, we show that $\zeta(s)$ has a meromorphic continuation to the complex plane and derive its functional equation.

Theorem 1 (Riemann). *Let*

$$\xi(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s), \quad \Re(s) > 1,$$

where $\Gamma(s)$ is the Euler gamma function. Then $\xi(s)$ has a meromorphic continuation to the whole complex plane. It is holomorphic on $\mathbb{C} \setminus \{0, 1\}$ and has simple pole at $s = 0, 1$ with residue $-1, 1$, respectively. Moreover, It satisfies the functional equation

$$\xi(s) = \xi(1 - s).$$

Proof. Recall the gamma function is defined as

$$\Gamma(s) := \int_0^\infty t^s e^{-t} \frac{dt}{t}, \quad \Re(s) > 0.$$

We substitute $s \mapsto s/2$, multiply the factor $\pi^{-s/2} n^{-s}$ on both sides, do some change of variables, and get

$$\begin{aligned}
\pi^{-\frac{s}{2}} n^{-s} \Gamma\left(\frac{s}{2}\right) &= \int_0^\infty \pi^{-\frac{s}{2}} n^{-s} t^{\frac{s}{2}} e^{-t} \frac{dt}{t} \\
&= \int_0^\infty \left(\frac{t}{\pi n^2}\right)^{\frac{s}{2}} e^{-t} \frac{dt}{t} \\
&= \int_0^\infty t^{\frac{s}{2}} e^{-\pi n^2 t} \frac{dt}{t} \qquad \left(\frac{t}{\pi n^2} \mapsto t\right).
\end{aligned}$$

Summing both sides from 1 to ∞ with respect to n, we get

$$\begin{aligned}\sum_{n=1}^{\infty} \pi^{-\frac{s}{2}} n^{-s} \Gamma\left(\frac{s}{2}\right) &= \sum_{n=1}^{\infty} \int_0^{\infty} t^{\frac{s}{2}} e^{-\pi n^2 t} \frac{dt}{t} \\ &= \frac{1}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \int_0^{\infty} t^{\frac{s}{2}} e^{-\pi n^2 t} \frac{dt}{t} \\ &= \int_0^{\infty} \left(\frac{\theta(it)-1}{2}\right) t^{\frac{s}{2}} \frac{dt}{t}.\end{aligned} \tag{2}$$

Here,

$$\theta(z) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}$$

is called the *Jacobi theta series*, which converges absolutely for $\Im(z) > 0$. Note that by the definition of $\xi(s)$, the left-hand side of (2) is

$$\sum_{n=1}^{\infty} \pi^{-\frac{s}{2}} n^{-s} \Gamma\left(\frac{s}{2}\right) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \xi(s).$$

So we get

$$\xi(s) = \int_0^{\infty} \left(\frac{\theta(it)-1}{2}\right) t^{\frac{s}{2}} \frac{dt}{t}. \tag{3}$$

In order to calculate the right-hand side of (3), we need the following functional equation of theta series.

Lemma 2. *The theta series $\theta(z)$ satisfies the functional equation*

$$\theta\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{i}}\theta(z), \quad \Im(z) > 0.$$

In particular,

$$\theta\left(\frac{i}{t}\right) = t^{\frac{1}{2}}\theta(it), \quad t > 0.$$

Proof. By identity theorem, it suffices to prove the second equality. Consider

$$f_t(x) := e^{-\pi t x^2}.$$

The Fourier transform of $f_t(x)$ is

$$\hat{f}_t(y) = t^{-\frac{1}{2}} e^{-\frac{\pi y^2}{t}}.$$

So by *Poisson summation formula*, we have

$$\theta(it) = \sum_{n\in\mathbb{Z}} f_t(n) = \sum_{n\in\mathbb{Z}} \hat{f}_t(n) = t^{-\frac{1}{2}}\theta\left(\frac{i}{t}\right).$$

□

Returning to the right-hand side of (3), we see that

$$\begin{aligned}
&\int_0^\infty \left(\frac{\theta(it)-1}{2}\right) t^{\frac{s}{2}} \frac{dt}{t} \\
&= \frac{1}{2}\int_0^1 (\theta(it)-1)t^{\frac{s}{2}}\frac{dt}{t} + \boxed{\frac{1}{2}\int_1^\infty (\theta(it)-1)t^{\frac{s}{2}}\frac{dt}{t}} \\
&= \frac{1}{2}\int_1^\infty \left(\theta\left(\frac{i}{t}\right)-1\right) t^{-\frac{s}{2}}\frac{dt}{t} + \boxed{} && \left(t\mapsto \frac{1}{t}\right) \\
&= \frac{1}{2}\int_1^\infty \left(t^{\frac{1}{2}}\theta(it)-1\right) t^{-\frac{s}{2}}\frac{dt}{t} + \boxed{} && \text{(by Lemma 2)} \\
&= \frac{1}{2}\int_1^\infty \left((\theta(it)-1)t^{\frac{1}{2}(1-s)} + t^{\frac{1}{2}(1-s)} - t^{-\frac{s}{2}}\right)\frac{dt}{t} + \boxed{} \\
&= \frac{1}{2}\int_1^\infty (\theta(it)-1)t^{\frac{1}{2}(1-s)}\frac{dt}{t} - \frac{1}{1-s} - \frac{1}{s} + \boxed{\frac{1}{2}\int_1^\infty (\theta(it)-1)t^{\frac{s}{2}}\frac{dt}{t}}.
\end{aligned}$$

So we get

$$\xi(s) = \frac{1}{2}\int_1^\infty (\theta(it)-1)t^{\frac{1}{2}(1-s)}\frac{dt}{t} + \frac{1}{2}\int_1^\infty (\theta(it)-1)t^{\frac{s}{2}}\frac{dt}{t} - \frac{1}{1-s} - \frac{1}{s}. \tag{4}$$

Note that the expression on the right is unchanged under the substitution $s\mapsto 1-s$. Thus, the same is true for the left-hand side. In other words, we have

$$\xi(s) = \xi(1-s).$$

Also, the integrals in (4) are holomorphic for all $s\in\mathbb{C}$. Hence, $\xi(s)$ is holomorphic on $\mathbb{C}\setminus\{0,1\}$ and has simple pole at $s=0,1$ with residue $-1,1$, respectively. This completes the proof. □

Corollary 3. *$\zeta(s)$ has a meromorphic continuation to $\mathbb{C}\setminus\{1\}$ and a simple pole at $s=1$ with residue 1. Moreover, it satisfies the functional equation*

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$

Proof. From Theorem 1, we multiply $\Gamma(1-s/2)$ on both sides and get

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\zeta(1-s). \tag{5}$$

Using the reflection formula of the gamma function:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z\notin\mathbb{Z},$$

the left-hand side of (5) becomes

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{s}{2}}\frac{\pi}{\sin\left(\frac{\pi s}{2}\right)}\zeta(s). \tag{6}$$

On the other hand, using Legendre's duplication formula of the gamma function:

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\pi^{\frac{1}{2}}\Gamma(2z),$$

the right hand side of (5) becomes

$$\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\zeta(1-s) = \pi^{-\frac{1-s}{2}}2^{1-2\left(\frac{1-s}{2}\right)}\pi^{\frac{1}{2}}\Gamma(1-s)\zeta(1-s). \tag{7}$$

We may now equate (6) and (7) and get

$$\pi^{-\frac{s}{2}}\frac{\pi}{\sin\left(\frac{\pi s}{2}\right)}\zeta(s) = \pi^{-\frac{1-s}{2}}2^{1-2\left(\frac{1-s}{2}\right)}\pi^{\frac{1}{2}}\Gamma(1-s)\zeta(1-s).$$

This simplifies to the desired functional equation. □

Let s be a negative even integer. In this case, the sine factor in Corollary 3 is zero while the others are non-zero. Thus, we have $\zeta(s) = 0$ for all $s \in \{-2n \mid n \in \mathbb{N}\}$. Such s is called a *trivial zero* of $\zeta(s)$. The famous Riemann hypothesis is a conjecture about the locations of all non-trivial zeros of $\zeta(s)$.

Conjecture 4 (Riemann Hypothesis)**.** All non-trivial zeros of $\zeta(s)$ lie on the line $\Re(s) = 1/2$.

Next, we express the Riemann zeta function at negative integers in terms of Bernoulli numbers. Recall Euler's calculation of Riemann zeta function at positive even integers.

Theorem 5 (Euler)**.** *For all positive even integer $n \in \mathbb{N}$, we have*

$$\zeta(n) = -\frac{B_n}{2 \cdot n!}(2\pi i)^n$$

where B_k is the Bernoulli number given by $t/(e^t - 1) = \sum_{k=0}^{\infty} B_k t^k/k!$.

Proof. See [Cha, Theorem 1.1]. □

Corollary 6. *For $k \in \mathbb{N}$ with $k > 1$, we have*

$$\zeta(1-k) = -\frac{B_k}{k}$$

where B_k is the k-th Bernoulli number.

Proof. By Corollary 3 and Theorem 5, when $k > 1$ is even, we have

$$\begin{aligned}\zeta(1-k) &= 2^{1-k}\pi^{-k}\sin\left(\frac{\pi(1-k)}{2}\right)\Gamma(k)\zeta(k)\\ &= 2^{1-k}\pi^{-k}(-1)^{\frac{k}{2}}(k-1)!\left(-\frac{B_k}{2 \cdot k!}(2\pi i)^k\right)\\ &= -\frac{B_k}{k}.\end{aligned}$$

When $k > 1$ is odd, we have $\zeta(1-k) = 0$ as $1-k$ is a trivial zero. One also recalls the fact that $B_k = 0$ for all odd $k > 1$ (see [Cha, Corollary 1.3]). This completes the proof. □

Remark 7. From Corollary 6, we take $k = 2$ and see that

$$\zeta(-1) = -\frac{B_2}{2} = -\frac{1}{12}.$$

Some people would interpret this identity as

$$\zeta(-1) = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$$

and claim that the sum of all natural numbers is $-1/12$. This is incorrect because the series expression (1) for $\zeta(s)$ is only valid for $\Re(s) > 1$.

References

[Cha] Timo Chang. *Special Zeta Values at Positive Even Integers (Classical Case and Function Field Case)*. Expository papers. URL: https://timomath.com/special_zeta_values_at_positive_even_integers_classical_case_and_function_field_case.pdf.