Simple Math Puzzles

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Contents

1 Covering the Plane with Lines

A set is called *finite* if it contains a finite number of elements—regardless of how large that number may be. Otherwise, it is considered *infinite*. Moreover, if there exists a bijection (a one-to-one correspondence) between an infinite set and the set of positive integers, the set is said to be *countable*; if no such bijection exists, the set is called *uncountable*.

Problem 1. Suppose you are drawing lines on the Cartesian plane with the goal of covering the entire plane. That is, every point in the plane must lie on at least one of your lines. Prove that this cannot be achieved using only countably many lines. In other words, to cover the entire plane with lines, you need uncountably many of them.

First, you can't cover the entire plane using only finitely many lines (Why?). On the other hand, if there is no restriction on the number of lines, then the problem becomes trivial: simply draw vertical lines through every point on the x-axis. But since the x-axis contains uncountably many points, this method necessarily involves uncountably many lines.

In our problem, however, you're limited to using only countably many lines. Why does this make it impossible to cover the entire plane? Let's begin the proof.

Proof. Suppose we have drawn countably many lines on the plane. Then by definition, we can label them as $L_1, L_2, L_3, \ldots$. Our goal is to show that there exists at least one point in the plane that does not lie on any of these lines.

As we mentioned earlier, there are uncountably many vertical lines in the plane. That means there are "more" vertical lines than the countably many lines $L_1, L_2, L_3, \ldots$ that we have drawn. Therefore, there must exist at least one vertical line that is not among them. Let's call such a line L'. Our desired point is located somewhere on L'.

We can categorize the lines $L_1, L_2, L_3, \ldots$ into two types:

- Those that are parallel to L', and so do not intersect L' at all.
- Those that are not parallel to L', each of which intersects L' at exactly one point.

Therefore, the total number of intersection points between L' and the lines $L_1, L_2, L_3, \ldots$ comes entirely from the second type, which is equal to the number of lines not parallel to L'. Since these lines are part of a countable set, their number must be either finite or countably infinite (i.e., *at most countable*). Thus, L' intersects the lines $L_1, L_2, L_3, \ldots$ in at most countably many points.

But since the number of points on L' is uncountable, this means there must exist at least one point on L' that is not on any of the lines $L_1, L_2, L_3, \ldots$. This completes the proof. □

2 Shifting Rooms Problem

Problem 2. Suppose we are given a finite number of rooms, each containing a finite number of people. At each step, one person must leave their current room and move to another. The rule is that a person may only move to a room that contains at least as many people as their current room. Prove that after a finite number of steps, all the people will end up in the same room.

If you take a moment to think about this problem, you are likely to convince yourself with little difficulty. Since the rule requires moving only to rooms that are equally or more populated, people will gradually leave the less populated rooms and join the more populated ones. Eventually, everyone will gather in the same room.

This idea may seem straightforward at first glance, but can you provide a rigorous proof? Let's begin with a simple observation. Every time someone moves from one room to another, the total number of people in all rooms remains the same. But what happens if we don't just sum the counts directly, and instead, sum their squares?

For simplicity, suppose there are only two rooms A and B, initially containing 2 and 3 people, respectively. Since Room A has fewer people, a person can only move from A to B. In this case, the configuration changes from $(2, 3)$ to $(1, 4)$. If we simply sum the number of people, the total remains the same:

$$2 + 3 = 1 + 4 = 5.$$

But if we instead square the number of people in each room and sum, it changes from

$$2^2 + 3^2 = 13$$

to

$$1^2 + 4^2 = 17.$$

We see that the sum increases.

With this idea in mind, we now give a formal proof.

Proof. Suppose there are n rooms $R_1, \dots, R_n$ containing $x_1, \dots, x_n$ people, respectively. As we observed earlier, we focus on the sum of the squares of the number of people in each room.

Consider a step where someone moves from room R_i to room R_j. According to the rule, this means room R_i contains no more people than room R_j. In other words,

$$x_i \leq x_j. \tag{1}$$

After the move, the number of people in rooms R_i and R_j changes from

$$(x_i, x_j)$$

to

$$(x_i - 1, x_j + 1),$$

and the corresponding square-sum changes from

$$x_i^2 + x_j^2 \tag{2}$$

to

$$(x_i - 1)^2 + (x_j + 1)^2. \tag{3}$$

We note that

$$(3) = (x_i - 1)^2 + (x_j + 1)^2 = x_i^2 - 2x_i + 1 + x_j^2 + 2x_j + 1 = x_i^2 + x_j^2 + 2(x_j - x_i) + 2.$$

By (1), the last two terms $2(x_j - x_i) + 2 \geq 2 > 0$. It follows that

$$(3) = (x_i - 1)^2 + (x_j + 1)^2 > x_i^2 + x_j^2 = (2).$$

This shows that the square-sum strictly increases with each move.

On the other hand, note that this sequence is bounded above by[1]

$$\left(\sum_{k=1}^{n} x_k \right)^2,$$

which corresponds to the square-sum when all people are in the same room. This bound is attainable in finitely many steps as the square-sum increases strictly with each move. Therefore, all the people will eventually end up in the same room. □

[1]Exercise: Given finitely many non-negative integers, prove that the sum of their squares is bounded above by the square of their total sum. That is, for any $a_1, \dots, a_n \in \mathbb{N} \cup \{0\}$, we have $a_1^2 + \cdots + a_n^2 \leq (a_1 + \cdots + a_n)^2$.