Special Zeta Values at Positive Even Integers (Classical Case and Function Field Case)

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Last edited: May 20, 2025

The theory of special zeta values has always been of great interest for mathematicians. Classically, it was established by Euler the values of Riemann zeta function at positive even integers. The formula he gave involves several quantities which are all of importance in their own ways, including the period $2\pi i$ (transcendental!). On the other hand, the study of zeta function in function field side was initiated by Carlitz in 1930s. He defined an analog of Riemann zeta function over $\mathbb{F}_q(\theta)$ (where θ is a variable) and gave a formula for its values at positive "even" integers. Surprisingly, and perhaps more excitingly, his formula matches perfectly to Euler's result.

This essay aims to present these two formulas in one place, and at the same time, give readers who are new to function field arithmetic a taste of how similar the classical and function fields worlds can be. As a matter of fact, the idea behind these two proofs runs also interestingly in parallel. In short, we exploit two expressions of certain functions and do some algebraic manipulation which lead to the desired result.

1 Classical Case

Recall the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\Re(s) > 1$. In this section we prove

Theorem 1.1 (Euler)**.** *For a positive even integer $n \in \mathbb{N}$, we have*

$$\zeta(n) = -\frac{B_n}{2 \cdot n!}(2\pi i)^n$$

where $B_k \in \mathbb{Q}$ is the Bernoulli number defined by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

In particular, this formula implies Riemann zeta function is transcendental at positive even integers. We note that the transcendence at positive odd integers is still widely open.

Proof. In the classical case, we exploit two expressions of the sine function. First, from the infinite product formula

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right),$$

we take log derivative and get

$$\pi \cot(\pi z) = \frac{\pi \cos(\pi z)}{\sin(\pi z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{-\frac{2z}{n^2}}{1 - \frac{z^2}{n^2}} = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n-z}\right). \tag{1}$$

On the other hand, from the exponential expressions of trigonometric functions

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i},$$

we have

$$\cot(z) = i \cdot \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} = i + \frac{2i}{e^{2iz} - 1}.$$

And so

$$\pi \cot(\pi z) = \pi i + \frac{2\pi i}{e^{2\pi i z} - 1}. \tag{2}$$

Equating (1) and (2), we get

$$\pi i + \frac{2\pi i}{e^{2\pi i z} - 1} = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n-z}\right).$$

And so

$$\pi i z + \frac{2\pi i z}{e^{2\pi i z} - 1} = 1 + \sum_{n=1}^{\infty} \left(\frac{2\pi i z}{2\pi i n + 2\pi i z} - \frac{2\pi i z}{2\pi i n - 2\pi i z}\right).$$

Let $t = 2\pi i z$, then we get

$$\frac{t}{2} + \frac{t}{e^t - 1} = 1 + \sum_{n=1}^{\infty} \left(\frac{t}{2\pi i n + t} - \frac{t}{2\pi i n - t}\right).$$

The left hand side is by definition,

$$\text{LHS} = \frac{t}{2} + \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

And the right hand side is

$$\begin{aligned}
\text{RHS} &= 1 + \sum_{n=1}^{\infty} \frac{t}{2\pi in} \left(\frac{1}{1+\frac{t}{2\pi in}} - \frac{1}{1-\frac{t}{2\pi in}} \right) \\
&= 1 + \sum_{n=1}^{\infty} \frac{t}{2\pi in} \left(\sum_{k=0}^{\infty} \left(-\frac{t}{2\pi in} \right)^k - \sum_{k=0}^{\infty} \left(\frac{t}{2\pi in} \right)^k \right) \\
&= 1 + \sum_{n=1}^{\infty} \frac{-2t}{2\pi in} \left(\sum_{k=0}^{\infty} \left(\frac{t}{2\pi in} \right)^{2k+1} \right) \\
&= 1 - 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{t}{2\pi in} \right)^{2k+2} \\
&= 1 - 2 \sum_{k=0}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{(2\pi in)^{2k+2}} \right) t^{2k+2}.
\end{aligned}$$

So we have the equality

$$\frac{t}{2} + \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = 1 - 2 \sum_{k=0}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{(2\pi in)^{2k+2}} \right) t^{2k+2}. \tag{3}$$

Now, comparing the coefficients of t^m for positive even integer m, we have

$$\frac{B_m}{m!} = -2 \sum_{n=1}^{\infty} \frac{1}{(2\pi in)^m} = \frac{-2}{(2\pi i)^m} \sum_{n=1}^{\infty} \frac{1}{n^m} = \frac{-2}{(2\pi i)^m} \cdot \zeta(m).$$

So finally,

$$\zeta(m) = -\frac{B_m}{2 \cdot m!} (2\pi i)^m.$$

□

Corollary 1.2. *As $i \in \mathbb{C} \setminus \mathbb{R}$ and $B_k \in \mathbb{Q}$, we see that for $n \geq 2$,*

$$\frac{\zeta(n)}{(2\pi i)^n} \in \mathbb{Q} \iff n \text{ is even.}$$

Corollary 1.3. *By comparing the coefficients of other terms in (3), we have (i) $B_0 = 1$ (ii) $B_1 = -1/2$ (iii) $B_k = 0$ for odd $k > 1$. (That is, the k-th Bernoulli number vanishes when $k > 1$ is odd.)*

2 Function Field Case

In this section we deal with the function field case. The whole story was due to the fascinating work of Carlitz in [Car35]. Let $A := \mathbb{F}_q[\theta]$ be the polynomial ring over a finite field

$\mathbb{F}_q$ of characteristic $p > 0$ and A_+ be its subset consisting of all monic polynomials. Let $k := \mathbb{F}_q(\theta)$ be the field of fractions of A, k_∞ be the completion of k at infinity with respect to the (non-archimedean) absolute value $|\cdot|_\infty$ normalized so that $|\theta|_\infty = q$, and $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure of k_∞.

By definition, $|a|_\infty := q^{\deg a} = \#(A/aA)$ for $a \in A$. This is viewed as an analog of the usual absolute value because classically, one has $|n| = \#(\mathbb{Z}/n\mathbb{Z})$ for $n \in \mathbb{Z}$. Hence, we keep in mind the following analogy:

$$\begin{array}{ccccccccc} \mathbb{N} & \subset & \mathbb{Z} & \subset & \mathbb{Q} & \subset & \mathbb{R} & \subset & \mathbb{C} \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ A_+ & \subset & A & \subset & k & \subset & k_\infty & \subset & \mathbb{C}_\infty \end{array}$$

Recall the classical Riemann zeta function runs through all positive integers. So with the above correspondence, we define

Definition 2.1 (Carlitz)**.** For $n \in \mathbb{N}$, put

$$\zeta_A(n) := \sum_{a \in A_+} \frac{1}{a^n}.$$

Since the absolute value of the general term, $|1/a^n|_\infty = q^{-n \deg a}$, goes to zero as $\deg a$ goes to infinity, we see that the sum converges to an element in k_∞ by non-archimedean property. (See [Pap23, Lemma 2.7.1] if necessary.)

In the previous section, we proved the formula of Riemann zeta function at positive *even* integers. It turns out that there's a perfectly analogous result for Carlitz zeta function.

Theorem 2.2 (Carlitz)**.** *For a positive A-even integer $n \in \mathbb{N}$ (i.e., $q - 1 \mid n$), we have*

$$\zeta_A(n) = \frac{BC_n}{\Pi_n}\tilde{\pi}^n = -\frac{BC_n}{(q-1)\cdot \Pi_n}\tilde{\pi}^n$$

where (i) $BC_n \in k$ is the Bernoulli-Carlitz number (analog of the Bernoulli number) (ii) Π_n is the Carlitz factorial (analog of the usual factorial function) (iii) $\tilde{\pi}$ is a Carlitz period which is transcendental (analog of $2\pi i$).

We will introduce all quantities in a minute. But before that, we see that just like the classical case, this formula implies Carlitz zeta function is transcendental at positive A-even integers. However, as opposed to the classical case, all the other Carlitz zeta values are actually known to be transcendental as well. This is a highly deep result in [Yu91].

The A-evenness of a positive integer refers to the following analogy: The usual even integer is by definition, an integer which is divisible by 2. This quantity can be thought of as the cardinality of unit group of $\mathbb{Z}$, which consists of two elements (signs) $\{\pm 1\}$. From this point of view, since there are $q-1$ "signs" in A (its unit group $A^\times = \mathbb{F}_q^\times$ consists of this many elements), we call a positive integer *A-even* if it is divisible by $q-1$. (So the second formula matches the classical result even more nicely.)

We now introduce all necessary background. To ease the exposition, we do not prove anything we mention and refer to suitable materials for interested readers. Recall that in the classical case, we used two expressions of a certain function (sine function to be exact) as a medium. Now in the function field case, this role will be played by *Carlitz exponential function*, which is an analog of the usual exponential function in positive characteristic. Below we give the definition and compare their analogies.

For $i \geq 0$, let D_i be the product of all monic polynomials in A of degree i. Then the *Carlitz exponential function* is defined as

$$\exp_C(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{D_i}. \tag{4}$$

It induces an exact sequence

$$0 \longrightarrow A \cdot \tilde{\pi} \longrightarrow \mathbb{C}_\infty \xrightarrow{\exp_C} \mathbb{C}_\infty \longrightarrow 0$$

for some $\tilde{\pi} \in \mathbb{C}_\infty$, called *Carlitz period*. This element is unique up to sign-multiplies[1] (i.e., $\mathbb{F}_q^\times$-multiples) because different choices of $\tilde{\pi}$ generate the same A-module. Moreover, it is a fact by [Wad41] that $\tilde{\pi}$ is transcendental over k. We mention that Carlitz exponential function surjects to the *additive* group $\mathbb{C}_\infty$. And from the above exact sequence, a non-archimedean version of Weierstrass factorization theorem now implies that

$$\exp_C(z) = z \prod_{0 \neq a \in A} \left(1 - \frac{z}{a\tilde{\pi}}\right). \tag{5}$$

We emphasize that this is another expression of Carlitz exponential function. For more information about this particular function, see the classic treatment such as [Gos96].

The analogy with the classical world goes as follows: The usual exponential function is defined as

$$\exp(z) := \sum_{i=0}^{\infty} \frac{z^i}{i!}. \tag{6}$$

And it induces an exact sequence

$$0 \longrightarrow \mathbb{Z} \cdot 2\pi i \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \longrightarrow 1$$

where the element $2\pi i \in \mathbb{C}$ can be viewed as a "period" (the kernel of $\exp(z)$ is all integral multiples of $2\pi i$). One immediately sees the analogy between $2\pi i$ and $\tilde{\pi}$. This also justifies the name Carlitz *period*. Since the elements $\pm 2\pi i$ generate the same integral multiples, so the period is really unique up to sign. Moreover, it is well known that $2\pi i$ is transcendental

[1] In Theorem 2.2 we've assumed $q-1 \mid n$, so $\tilde{\pi}^n$ is really well-defined.

over $\mathbb{Q}$. But we point out that the classical exponential function surjects to the *multiplicative* group $\mathbb{C}^\times$. This is one of the main difference between these two worlds.

These two functions also suggest the notion of Carlitz factorial. For a non-negative integer n, write $n = \sum_{i=0}^{m} n_i q^i$ in q-adic expansion, i.e., $0 \leq n_i \leq q-1$ for all i. Then we define the *Carlitz factorial* as

$$\Pi_n := \prod_{n=0}^{m} D_i^{n_i}.$$

Note that in (4), the denominator $D_i = \Pi_{q^i}$ is the Carlitz factorial of the corresponding exponent, whereas in (6), the denominator $i!$ is also the usual factorial of the corresponding exponent.

Lastly, recall the classical Bernoulli number $B_n \in \mathbb{Q}$ is given by the power series

$$\frac{z}{\exp(z) - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

With the nice analogs of exponential and factorial, we may now define the *Bernoulli-Carlitz number* $BC_n \in k$ to be

$$\frac{z}{\exp_C(z)} = \sum_{n=0}^{\infty} \frac{BC_n}{\Pi_n} z^n.$$

The difference between two denominators on the left is because in the classical world, 1 is the identity of the multiplicative group $\mathbb{C}^\times$, whereas in the function field world, 0 is the identity of the additive group $\mathbb{C}_\infty$. We refer to [Gos96, Chapter 9.2] for a table of Bernoulli-Carlitz numbers in the case $q = 3$.

We have prepared everything we need to prove the main theorem.

Proof. From the infinite product formula of Carlitz exponential function (5)

$$\exp_C(z) = z \prod_{0 \neq a \in A} \left(1 - \frac{z}{a\tilde{\pi}}\right),$$

we take "log derivative" (by which we mean for any power series f over $\mathbb{C}_\infty$, the quotient f'/f) and get

$$\frac{\exp_C'(z)}{\exp_C(z)} = \frac{1}{z} + \sum_{0 \neq a \in A} \frac{-\frac{1}{a\tilde{\pi}}}{1 - \frac{z}{a\tilde{\pi}}} = \frac{1}{z} + \sum_{0 \neq a \in A} \frac{1}{z - a\tilde{\pi}} = \frac{1}{z} + \sum_{0 \neq a \in A} \frac{1}{z + a\tilde{\pi}},$$

where the last equality is because the set $\{-a \mid 0 \neq a \in A\}$ is identical to $A \setminus \{0\}$. So we have

$$z \cdot \frac{\exp_C'(z)}{\exp_C(z)} = 1 + \sum_{0 \neq a \in A} \frac{z}{z + a\tilde{\pi}}.$$

Note that from the infinite summation expression of Carlitz exponential function (4)

$$\exp_C(z) = z + \sum_{m=1}^{\infty} \frac{z^{q^m}}{D_m},$$

we have $\exp'_C(z) = 1$. (Remind you that we are now working in the characteristic p world.) So the left hand side is by definition,

$$\text{LHS} = \frac{z}{\exp_C(z)} = \sum_{n=0}^{\infty} \frac{BC_n}{\Pi_n} z^n. \tag{7}$$

On the other hand, one sees that the right hand side is

$$\begin{aligned} \text{RHS} &= 1 + \sum_{0 \neq a \in A} \frac{z}{a\tilde{\pi}} \cdot \frac{1}{1 + \frac{z}{a\tilde{\pi}}} \\ &= 1 + \sum_{0 \neq a \in A} \frac{z}{a\tilde{\pi}} \sum_{m=0}^{\infty} \left(-\frac{z}{a\tilde{\pi}}\right)^m \\ &= 1 + \sum_{m=0}^{\infty} \frac{(-1)^m}{\tilde{\pi}^{m+1}} \left(\sum_{0 \neq a \in A} \frac{1}{a^{m+1}} \right) z^{m+1}. \end{aligned}$$

Here the change of order of two summations in a non-archimedean world is justified by [Pap23, Exercise 2.7.2]. Note that for each positive integer $s \in \mathbb{N}$,

$$\sum_{0 \neq a \in A} \frac{1}{a^s} = \sum_{\epsilon \in \mathbb{F}_q^\times} \frac{1}{\epsilon^s} \sum_{a \in A_+} \frac{1}{a^s} = \zeta_A(s) \sum_{\epsilon \in \mathbb{F}_q^\times} \epsilon^s,$$

where the last sum is seen to be -1 if $q-1$ divides s and 0 otherwise. So the right hand side becomes

$$\text{RHS} = 1 + \sum_{\substack{m=0 \\ q-1 \mid m+1}}^{\infty} \frac{(-1)^{m+1}}{\tilde{\pi}^{m+1}} \zeta_A(m+1) z^{m+1} = 1 + \sum_{\substack{n=1 \\ q-1 \mid n}}^{\infty} \frac{(-1)^n}{\tilde{\pi}^n} \zeta_A(n) z^n. \tag{8}$$

Equating (7) and (8), we get

$$\sum_{n=0}^{\infty} \frac{BC_n}{\Pi_n} z^n = 1 + \sum_{\substack{n=1 \\ q-1 \mid n}}^{\infty} \frac{(-1)^n}{\tilde{\pi}^n} \zeta_A(n) z^n. \tag{9}$$

Comparing the coefficients of z^n for positive A-even integer n, we have

$$\frac{BC_n}{\Pi_n} = \frac{(-1)^n}{\tilde{\pi}^n} \zeta_A(n).$$

Note that if the characteristic p is odd, then $q-1$ is even and thus so is n. And if p is even, then $-1 = 1$. In either case the power of -1 can be omitted. So finally,

$$\zeta_A(n) = \frac{BC_n}{\Pi_n} \tilde{\pi}^n.$$

□

It is a fact that $\tilde{\pi} \in (-\theta)^{1/(q-1)} \cdot k_\infty^\times$. In particular, this implies $\tilde{\pi}^n \in k_\infty \iff q-1 \mid n$. So by Theorem 2.2, we have the following corollary. (Compare it with Corollary 1.2.)

Corollary 2.3. *For $n \in \mathbb{N}$,*

$$\frac{\zeta_A(n)}{\tilde{\pi}^n} \in k \iff n \text{ is } A\text{-even.}$$

We also have an analog result of Corollary 1.3 by comparing the coefficients of other terms in (9).

Corollary 2.4. *(i) $BC_0 = 1$ (ii) $BC_n = 0$ if $q-1 \nmid n$. (That is, the n-th Bernoulli-Carlitz number vanishes when n is A-odd.)*

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