The Transcendence of e

Timo Chang
timo65537@protonmail.com

Last edited: January 31, 2025

In basic Calculus class, it is a standard exercise showing the number e is irrational by using the series expression $e = \sum_{n=0}^{\infty} 1/n!$. Although this had already been established by Euler in 1744, it was not known whether e is algebraic or not until Hermite's work in 1873 (over a hundred years later). In this essay, we demonstrate an elementary, and yet highly ingenious proof given by Hurwitz, that e is transcendental. The tool it will be used involves nothing but basic Calculus.

Theorem 1. *e is transcendental.*

Proof. We first make a general observation. For any polynomial $f(x)$ of degree r, put

$$F(x) := f(x) + f'(x) + \cdots + f^{(r)}(x).$$

We note that

$$\begin{aligned}(e^{-x}F(x))' &= e^{-x}\left(-F(x) + F'(x)\right) \\ &= e^{-x}\left(-f(x) + f^{(r+1)}(x)\right) \\ &= -e^{-x}f(x)\end{aligned}$$

where the last equality is because $\deg f = r$. So by mean value theorem we have for each $n \in \mathbb{N}$ that $e^{-n}F(n) - e^0F(0) = -e^{-\xi_n}f(\xi_n) \cdot (n-0)$ where ξ_n is between 0 and n. This implies

$$F(n) - e^nF(0) = -ne^{n-\xi_n}f(\xi_n) =: \epsilon_n. \tag{1}$$

Now, we assume by contradiction that e is algebraic. Then we have the equation

$$c_Ne^N + \cdots + c_1e + c_0 = 0$$

for some $c_n \in \mathbb{Z}$. This together with (1) gives

$$\begin{aligned}\sum_{n=1}^{N} c_n \epsilon_n &= \sum_{n=1}^{N} c_n (F(n) - e^n F(0)) \\ &= \sum_{n=1}^{N} c_n F(n) - F(0) \sum_{n=1}^{N} c_n e^n \\ &= \sum_{n=1}^{N} c_n F(n) - F(0) \cdot (-c_0).\end{aligned}$$

Hence we obtain the identity

$$\sum_{n=1}^{N} c_n \epsilon_n = c_0 F(0) + \sum_{n=1}^{N} c_n F(n). \tag{2}$$

Note that this holds for any polynomial f. Our goal is to choose f in a clever way so that it leads to a contradiction. More precisely, for any prime number p, we set

$$f(x) := \frac{1}{(p-1)!} \cdot x^{p-1}(1-x)^p(2-x)^p \cdots (N-x)^p, \tag{3}$$

and as before, $F(x) := f(x) + f'(x) + \cdots + f^{(r)}(x)$ where $r = \deg f$. Our strategy goes as follows: We claim that there exists a large prime number p so that for this particular f, we have in equation (2),

1. The left hand side is an integer divisible by p.
2. $c_0 F(0)$ on the right is an integer not divisible by p.
3. $c_n F(n)$ on the right is an integer divisible by p for each $n = 1, \ldots, N$.

And this will lead to a contradiction.

We manage these claims in reverse order. For the third one, recall that each c_n is an integer. So it's sufficient to show that

$$F(n) \text{ is an integer divisible by } p \text{ for all } n = 1, \ldots, N.$$

And recall by definition, $F(x) := f(x) + f'(x) + \cdots + f^{(r)}(x)$. So we show that $f^{(i)}(n)$ is an integer divisible by p for all $i = 0, 1, \ldots, r$ and such n.

- Case 1: $0 \leq i \leq p-1$. In this case one sees immediately by (3) that $f^{(i)}(n) = 0$ for all $n = 1, \ldots, N$ as each such n is a root of f with multiplicity p.

- Case 2: $p \leq i \leq r$. We justify the case by showing that $f^{(i)}$ is a polynomial with integral coefficients divisible by p. Note that it's enough to consider the terms of f with degree not less than i. So from (3) we write

  $$f(x) = \cdots + \sum_{j \in \mathbb{Z}_{\geq 0}} \frac{*}{(p-1)!} \cdot x^{i+j}$$

  where each $* \in \mathbb{Z}$. Then

  $$f^{(i)}(x) = \sum_{j \in \mathbb{Z}_{\geq 0}} * \cdot \frac{(i+j)(i+j-1)\cdots(j+1)}{(p-1)!} \cdot x^j.$$

  And we see that

  $$\begin{aligned} \frac{(i+j)(i+j-1)\cdots(j+1)}{(p-1)!} &= \frac{(i+j)!}{(p-1)!j!} \\ &= \frac{(i+j)!(p)(p+1)\cdots(i)}{i!j!} \\ &= \binom{i+j}{i}(p)(p+1)\cdots(i) \end{aligned}$$

  is in fact an integer divisible by p. (Note where the assumption on i is used.) This completes the case, and the third claim as well.

Next, we consider the second claim and show that

$$F(0) \text{ is an integer not divisible by } p.$$

Assume this for a moment. Then to finish the goal, we simply choose $p > c_0$ so that c_0 is not divisible by p.

For the claim, recall again $F(x) := f(x) + f'(x) + \cdots + f^{(r)}(x)$. We will show that $f^{(i)}(0)$ is an integer for all $i = 0, 1, \ldots, r$, and is not divisible by p if and only if $i = p - 1$.

- Case 1: $0 \leq i \leq p-2$. This is similar to the first case in the previous step. One sees from (3) that $f^{(i)}(0) = 0$ as 0 is a root of f with multiplicity $p-1$.
- Case 2: $i = p-1$. Note that $f^{(p-1)}(0)$ is the constant term of $f^{(p-1)}$, which is seen to be $(N!)^p$ by (3). So if we choose $p > N$, then it will not be divisible by p.
- Case 3: $p \leq i \leq r$. This follows from the observation made in the second case of the previous step that for each such i, $f^{(i)}$ is a polynomial with integral coefficients divisible by p.

Finally, we consider the first claim. Recall in (1), we have $\epsilon_n := -ne^{n-\xi_n} f(\xi_n)$ where ξ_n is between 0 and n. A very loose estimate shows that for each $n = 1, \ldots, N$ (so that

$0 \leq \xi_n \leq n \leq N$),

$$
\begin{aligned}
|\epsilon_n| &:= n e^{n-\xi_n} \cdot \frac{1}{(p-1)!} \xi_n^{p-1} |1-\xi_n|^p |2-\xi_n|^p \cdots |N-\xi_n|^p \\
&\leq N e^N \cdot \frac{1}{(p-1)!} N^{p-1} \underbrace{N^p \cdots N^p}_{N \text{terms}} \\
&= \frac{e^N (N^{N+1})^p}{(p-1)!} \to 0 \text{ as } p \to \infty
\end{aligned}
$$

by a standard fact concerning the limits: The factorial function grows eventually faster than the exponential function. Consequently, each ϵ_n can be arbitrarily small. So we choose p large enough so that

$$
\left| \sum_{n=1}^{N} c_n \epsilon_n \right| \leq \sum_{n=1}^{N} |c_n||\epsilon_n| < 1.
$$

That is, the absolute value of left hand side of (2) is less than 1. Since we've shown in the previous two steps that the right hand side is an integer, this forces the sum to be 0. So the claim is established. □