The Transcendence of e

Timo Chang
timo65537@protonmail.com

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In basic Calculus class, it is a standard exercise showing the number e is irrational by using the series expression $e = \sum_{n=0}^{\infty} 1/n!$. Although this had already been established by Euler in 1744, it was not known whether e is algebraic or not until Hermite's work in 1873 (over a hundred years later). In this essay, we demonstrate an elementary, and yet highly ingenious proof given by Hurwitz, that e is transcendental. The tool it will be used involves nothing but basic Calculus.

Theorem 1. *e is transcendental.*

Proof. We first make a general observation. For any polynomial $f(x)$ of degree r, put

$$F(x) := f(x) + f'(x) + \cdots + f^{(r)}(x).$$

We note that

$$\begin{aligned}(e^{-x}F(x))' &= e^{-x}\left(-F(x) + F'(x)\right) \\ &= e^{-x}\left(-f(x) + f^{(r+1)}(x)\right) \\ &= -e^{-x}f(x),\end{aligned}$$

where the last equality is because $\deg f = r$. So by mean value theorem, we have for each $n \in \mathbb{N}$ that $e^{-n}F(n) - e^0F(0) = -e^{-\xi_n}f(\xi_n) \cdot (n-0)$ where ξ_n is between 0 and n. This implies

$$F(n) - e^nF(0) = -ne^{n-\xi_n}f(\xi_n) =: \epsilon_n. \tag{1}$$

Now, we assume by contradiction that e is algebraic. Then we have the equation

$$c_Ne^N + \cdots + c_1e + c_0 = 0$$

for some $c_n \in \mathbb{Z}$. This together with (1) gives

$$\begin{aligned}\sum_{n=1}^{N} c_n\epsilon_n &= \sum_{n=1}^{N} c_n(F(n) - e^n F(0)) \\ &= \sum_{n=1}^{N} c_n F(n) - F(0)\sum_{n=1}^{N} c_n e^n \\ &= \sum_{n=1}^{N} c_n F(n) - F(0)\cdot(-c_0).\end{aligned}$$

Hence we obtain the identity

$$\sum_{n=1}^{N} c_n\epsilon_n = c_0F(0) + \sum_{n=1}^{N} c_nF(n). \tag{2}$$

Note that this holds for any polynomial f. Our goal is to choose f in a clever way so that it leads to a contradiction. More precisely, for any prime number p, we set

$$f(x) := \frac{1}{(p-1)!}\cdot x^{p-1}(1-x)^p(2-x)^p\cdots(N-x)^p, \tag{3}$$

and as before, $F(x) := f(x) + f'(x) + \cdots + f^{(r)}(x)$ where $r = \deg f$. Our strategy goes as follows: we claim that there exists a large prime number p so that for this particular f, we have in equation (2),

1. the left-hand side is an integer divisible by p.
2. $c_0F(0)$ on the right is an integer not divisible by p.
3. $c_nF(n)$ on the right is an integer divisible by p for each $n = 1,\dots,N$.

And this will lead to a contradiction.

We manage these claims in reverse order. For the third one, recall that each c_n is an integer. So it's sufficient to show that

$$F(n) \text{ is an integer divisible by } p \text{ for all } n = 1,\dots,N.$$

And recall by definition, $F(x) := f(x) + f'(x) + \cdots + f^{(r)}(x)$. So we show that $f^{(i)}(n)$ is an integer divisible by p for all $i = 0,1,\dots,r$ and $n = 1,\dots,N$.

- Case 1: $0 \le i \le p-1$. In this case, one sees immediately by (3) that $f^{(i)}(n) = 0$ for all $n = 1,\dots,N$ as each such n is a root of f with multiplicity p.

- Case 2: $p \le i \le r$. We justify the case by showing that $f^{(i)}$ is a polynomial with integral coefficients divisible by p. Note that it's enough to consider the terms of f with degree not less than i. So from (3), we write

$$f(x) = \cdots + \sum_{j \in \mathbb{Z}_{\ge 0}} \frac{*}{(p-1)!} \cdot x^{i+j},$$

where each $* \in \mathbb{Z}$. Then

$$f^{(i)}(x) = \sum_{j \in \mathbb{Z}_{\ge 0}} * \cdot \frac{(i+j)(i+j-1)\cdots(j+1)}{(p-1)!} \cdot x^j,$$

and we see that

$$\begin{aligned} \frac{(i+j)(i+j-1)\cdots(j+1)}{(p-1)!} &= \frac{(i+j)!}{(p-1)!j!} \\ &= \frac{(i+j)! \cdot p(p+1)\cdots i}{i!j!} \\ &= \binom{i+j}{i} \cdot p(p+1)\cdots i \end{aligned}$$

is in fact an integer divisible by p. (Note where the assumption on i is used.) This completes the case, and the third claim as well.

Next, we consider the second claim and show that

$$F(0) \text{ is an integer not divisible by } p.$$

Assume this for a moment. Then to finish the claim, we simply choose $p > c_0$ so that c_0 is not divisible by p.

For the claim, recall again $F(x) := f(x) + f'(x) + \cdots + f^{(r)}(x)$. We will show that $f^{(i)}(0)$ is an integer for all $i = 0, 1, \dots, r$, and is not divisible by p if and only if $i = p - 1$.

- Case 1: $0 \le i \le p-2$. This is similar to the first case in the previous step. One sees from (3) that $f^{(i)}(0) = 0$ as 0 is a root of f with multiplicity $p-1$.

- Case 2: $i = p-1$. Note that $f^{(p-1)}(0)$ is the constant term of $f^{(p-1)}$, which is seen to be $(N!)^p$ by (3). So if we choose $p > N$, then it will not be divisible by p.

- Case 3: $p \le i \le r$. This follows from the observation made in the second case of the previous step that for each such i, $f^{(i)}$ is a polynomial with integral coefficients divisible by p.

Finally, we consider the first claim. Recall in (1), we have $\epsilon_n := -ne^{n-\xi_n} f(\xi_n)$ where ξ_n is between 0 and n. A very loose estimate shows that for each $n = 1, \dots, N$ (so that

$0 \le \xi_n \le n \le N$),

$$\begin{aligned}
|\epsilon_n| &:= ne^{n-\xi_n} \cdot \frac{1}{(p-1)!}\xi_n^{p-1}|1-\xi_n|^p|2-\xi_n|^p \cdots |N-\xi_n|^p \\
&\le Ne^N \cdot \frac{1}{(p-1)!}N^{p-1}\underbrace{N^p \cdots N^p}_{N\text{terms}} \\
&= \frac{e^N(N^{N+1})^p}{(p-1)!} \to 0 \text{ as } p \to \infty
\end{aligned}$$

by a standard fact concerning the limits: the factorial function grows eventually faster than the exponential function. Consequently, each ϵ_n can be arbitrarily small. So we choose p large enough so that

$$\left|\sum_{n=1}^{N} c_n\epsilon_n\right| \le \sum_{n=1}^{N}|c_n||\epsilon_n| < 1.$$

That is, the absolute value of left-hand side of (2) is less than 1. Since we've shown in the previous two steps that the right-hand side is an integer, this forces the sum to be 0. So the claim is established. □