Visual Proofs

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Contents

1 Determine the Area by Determinant

Back in high school, we learned that the area of the parallelogram spanned by two non-parallel vectors on the Cartesian plane can be computed using the determinant. In this section, we provide a visual explanation of this property.

Proposition 1.1. *Given two vectors $\vec{v}_1 = (a, b)$ and $\vec{v}_2 = (c, d)$ in the plane, the parallelogram $ABDC$ spanned by these two vectors (see Figure 1.1) has area*

$$\left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = |ad - bc|.$$

Proof. First, we assign coordinates to Figure 1.1 with point A at the origin. Then points B, C, D have coordinates $(a, b), (c, d), (a + c, b + d)$, respectively. Next, we draw an upright rectangle inside the parallelogram $ABDC$, as shown in Figure 1.2.

Since the rectangle is drawn upright, we can identify the coordinates of its vertices. In particular, one sees that the two dark-colored triangles (top and bottom) have equal area, and the same goes for the two light-colored triangles (left and right). Notice that the area of the parallelogram $ABDC$ is the area of the central rectangle plus the total area of the four triangles.

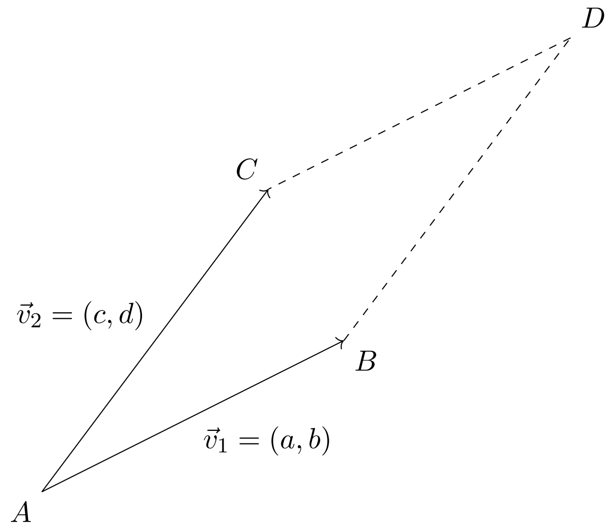

Figure 1.1

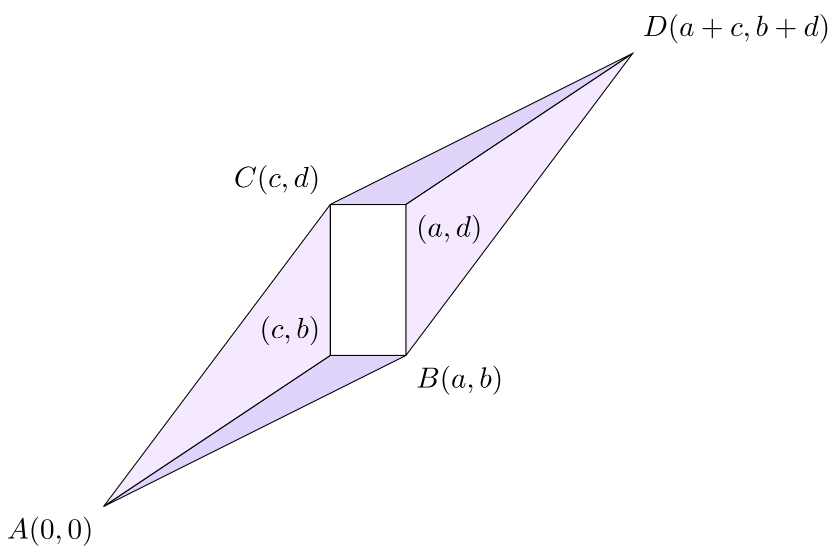

Figure 1.2

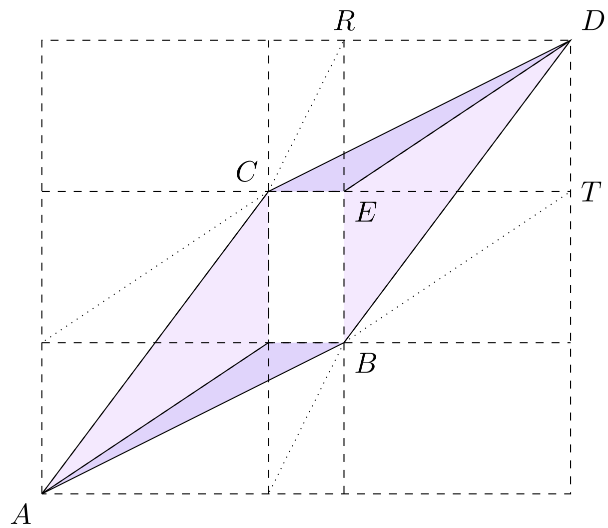

Figure 1.3

Next, we extend the four sides of the rectangle to form a dashed 3×3 grid around the parallelogram, draw four dotted lines, and label some points, as shown in Figure 1.3.

Let's look at the dark-colored triangle at the top. We have

$$\text{Area of } \triangle CED = \frac{1}{2} \times |\overline{CE}| \times |\overline{DT}|.$$

Notice that $|\overline{DT}| = |\overline{RE}|$, so $\triangle CER$ and $\triangle CED$ have the same height. Since they also share the same base $\overline{CE}$, their areas must be equal. Thus, we can "move" the top dark triangle from $\triangle CED$ to $\triangle CER$ without changing the area.

Applying the same transformation to the other three triangles (Exercise!), we arrive at the configuration shown in Figure 1.4.

It is now easy to see that the triangles of the same color can be combined into full rectangles. We also relabel their coordinates in Figure 1.2 before, as shown in Figure 1.5.

Recall that the area of the parallelogram $ABDC$ was the area of the central rectangle (which remained unchanged) plus the total area of the four triangles. Therefore, the area of the parallelogram is $\mathcal{A} + \mathcal{B} + \mathcal{D}$. We can now compute that

$$\mathcal{A} + \mathcal{B} + \mathcal{D} = (\mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D}) - \mathcal{C} = ad - bc.$$

This completes the proof. □

Remark 1.2. If the vectors are placed in the opposite order, we'll get a negative sign. This is why an absolute value is necessary in Proposition 1.1.

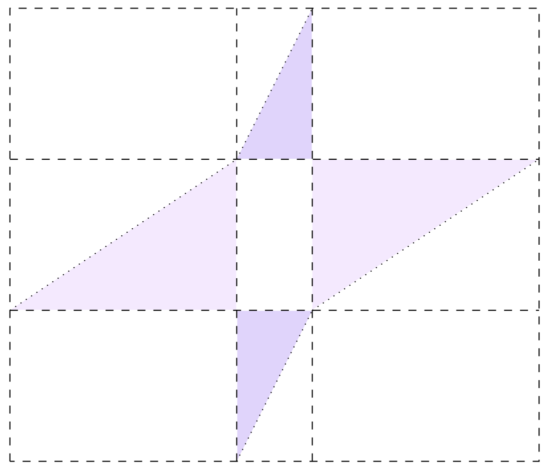

Figure 1.4

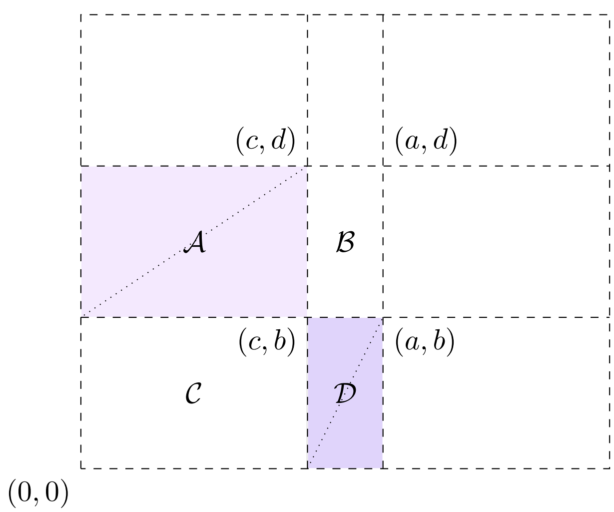

Figure 1.5

2 Pythagorean Theorem

Pythagorean theorem, probably the most well-known theorem among math amateurs, has already been discovered over 300 proofs. In this section, we demonstrate the one given by James Garfield, who was the 20th president of the United States. It is extremely simple and yet so elegant that any elementary school student can understand.

Let us first state the theorem for completeness.

Theorem 2.1 (Pythagorean Theorem)**.** *Consider the right triangle in Figure 2.1. The hypotenuse has length a, and the two legs have length b and c. Then we have*

$$a^2 = b^2 + c^2.$$

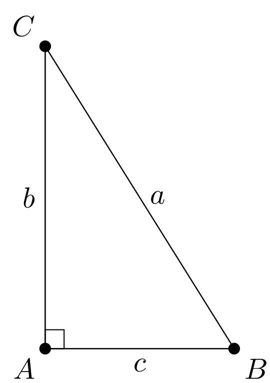

Figure 2.1

Proof. The idea of Garfield's marvelous proof is to find the area of a certain trapezoid in two different ways and create a useful equality. We duplicate our right triangle $\triangle ABC$, rotate it by 90° clockwise, and call it $\triangle A'B'C'$. Note that $\triangle ABC \cong \triangle A'B'C'$. We next join the points B' and C, as shonw in Figure 2.2.

Let's take a closer look at what's inside.

- Since $\triangle ABC \cong \triangle A'B'C'$, they must share the same area. Let's call it $\mathcal{A}$.
- We have $\angle ABC = \angle A'B'C'$, which is denoted it as α. And $\angle ACB = \angle A'C'B'$, which is denoted it as β. Since the interior angles of a triangle add up to 180°, we know $\alpha + \beta = 90°$.
- We draw a dashed line connecting B and C', and form a new triangle $\triangle BCC'$. Its area is denoted by $\mathcal{B}$.
- Since $\alpha + \beta = 90°$, we have $\angle BCC' = 90°$. Hence, $\triangle BCC'$ is a right triangle. And both legs have length a.

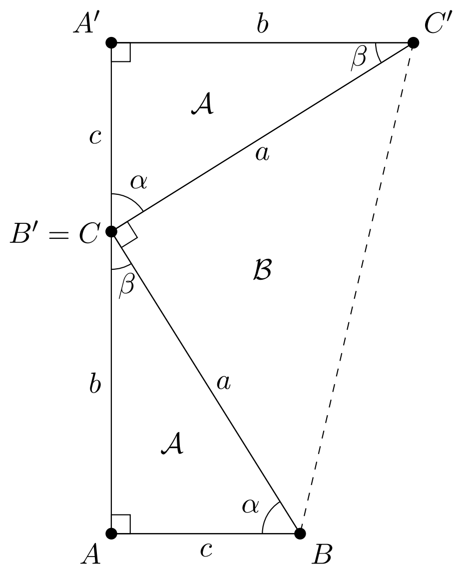

Figure 2.2

We have obtained a trapezoid $ABC'A'$. Let us compute its area in two different ways. First, using the formula of trapezoids, we know its area is

$$\frac{(b+c)\cdot(b+c)}{2}. \tag{1}$$

On the other hand, its area is also equal to $\mathcal{A}+\mathcal{A}+\mathcal{B}$. Using the formula of triangles, we know they are

$$\frac{bc}{2}+\frac{bc}{2}+\frac{a^2}{2}. \tag{2}$$

Since (1) and (2) are the same, we have

$$\frac{(b+c)\cdot(b+c)}{2}=\frac{bc}{2}+\frac{bc}{2}+\frac{a^2}{2}.$$

Clearing the denominators out and expanding the left-hand side give us

$$b^2+2bc+c^2=2bc+a^2.$$

Hence,

$$b^2+c^2=a^2.$$

This completes the proof. □

3 Sum of First n Squares

Every high school student should be familiar with this famous summation formula.

Proposition 3.1. *For any* $n \in \mathbb{N}$, *we have*

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Maybe you still remember how to "verify" this equation using mathematical induction, but none of your teacher has told you how to derive this formula from scratch. In this section, we give a simple explanation of this by using only manipulation of shapes.

Proof. We take $n = 5$ as an example. The general case will follow similarly. Note that finding $1^2+2^2+3^2+4^2+5^2$ is exactly the same as finding the total area of five squares with sides 1 to 5. Motivated by this, we draw three copies of them with one colored in advance, as Figure 3.1.

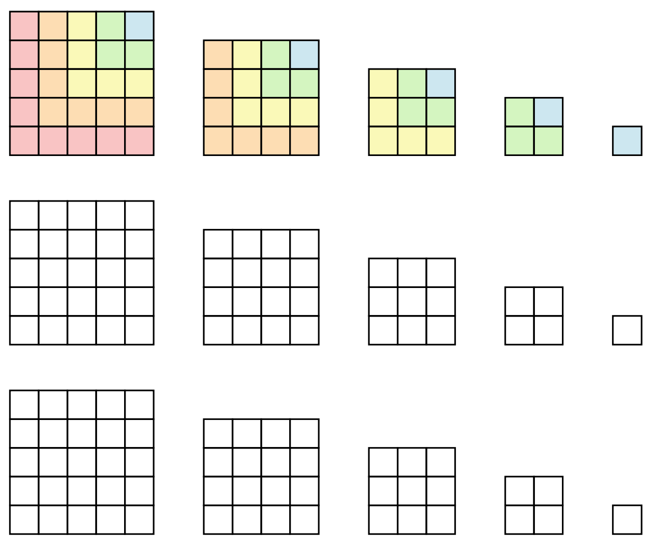

Figure 3.1

Next, we cut these squares and put them together, as shown in Figure 3.2. Pieces with colors are assembled into a tower. White squares with bold sides are those without any coloring.

Since we are just cutting and pasting, the area of this big rectangle must remain the same as those 15 squares in Figure 3.1, which is

$$3 \cdot (1^2 + 2^2 + 3^2 + 4^2 + 5^2). \tag{3}$$

On the other hand, let us look at Figure 3.2 very carefully. Its vertical side has length $1+2+3+4+5$. And the horizontal side has length equal to the number of pieces of small red squares adding 2. If we observe the top-left square in Figure 3.1, we see that there are $5+5-1$ such red squares. So the length of the horizontal side is $(5+5-1)+2 = 2 \cdot 5 + 1$.

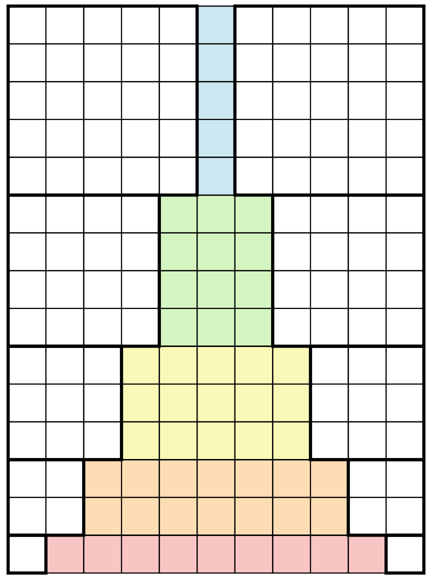

Figure 3.2

Thus, the area of the big rectangle in Figure 3.2 is equal to

$$(1+2+3+4+5)\cdot(2\cdot 5+1)=\frac{5(5+1)}{2}\cdot(2\cdot 5+1). \tag{4}$$

Here, we used the formula of sum of n consecutive integers.

Since both (3) and (4) represent the area in Figure 3.2, they must be equal. That is,

$$3\cdot(1^2+2^2+3^2+4^2+5^2)=\frac{5(5+1)}{2}\cdot(2\cdot 5+1).$$

Dividing both sides by 3 gives us

$$1^2+2^2+3^2+4^2+5^2=\frac{5(5+1)(2\cdot 5+1)}{6}.$$

And here it is, our desired formula. □

4 Sum of First n Cubes

In the previous section, we saw how to visualize the formula of sum of first n squares. Let us now do the same thing for the sum of first n cubes.

Proposition 4.1. *For any $n\in\mathbb{N}$, we have*

$$1^3+2^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2.$$

Proof. We again take $n = 5$ as an example, leaving the general case to the readers. Note that we may write

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 1 \cdot 1^2 + 2 \cdot 2^2 + 3 \cdot 3^2 + 4 \cdot 4^2 + 5 \cdot 5^2.$$

Thus, finding $1^3 + 2^3 + 3^3 + 4^3 + 5^3$ is exactly the same as finding the total area of one 1×1 square, two 2×2 squares, and so on, up to five 5×5 squares. Motivated by this idea, we draw these squares with colors and put them as in Figure 4.1.

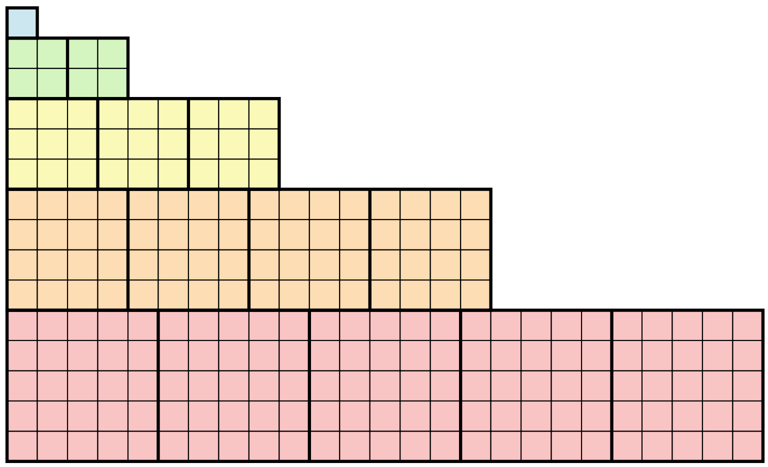

Figure 4.1

Next, we extend the bottom edge to the right by 5 units, and draw an auxiliary dashed line from the top-left corner to the far right end, as in Figure 4.2.

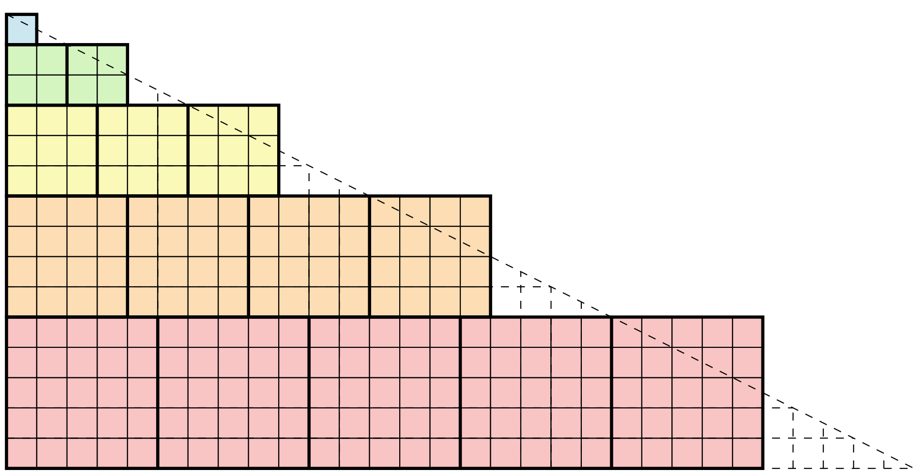

Figure 4.2

In each colored rectangle, we "move" the part above the auxiliary line to fill in the white dashed triangle below it, as shown in Figure 4.3.

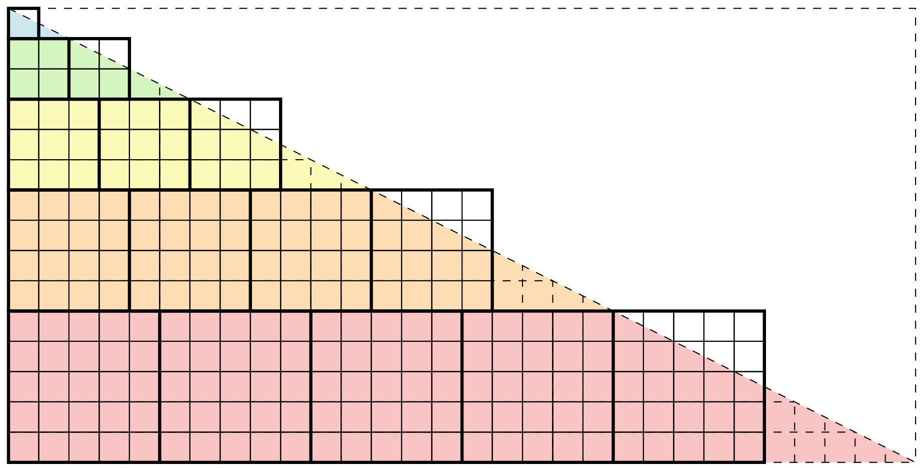

Figure 4.3

In this way, we see that the colored area is half of the rectangle with verticle side equals

$$1 + 2 + 3 + 4 + 5 = \frac{5(5+1)}{2}$$

(by the formula of sum of n consecutive integers) and horizontal side

$$5 \cdot 5 + 5 = 5(5+1).$$

This means that the colored area is

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = \frac{1}{2} \cdot \frac{5(5+1)}{2} \cdot 5(5+1) = \left(\frac{5(5+1)}{2}\right)^2,$$

which is our desired formula. □