When Proofs Go Wrong

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1 Every Triangle is Isosceles

Claim 1. Every triangle is isosceles.

Proof. Consider an arbitrary triangle $\triangle ABC$. We draw the angle bisector of $\angle A$ and the perpendicular bisector of $\overline{BC}$, which intersect at point P. Let Q be the midpoint of $\overline{BC}$. Next, we draw a line through P that is perpendicular to $\overline{AB}$ and intersects it at point E, and another line through P that is perpendicular to $\overline{AC}$ and intersects it at point F. Finally, we draw the segments $\overline{BP}$ and $\overline{PC}$. Figure 1.1 illustrates all information.

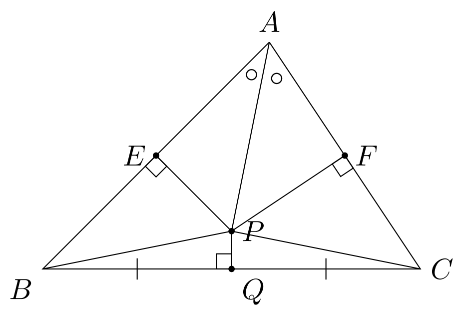

Figure 1.1

Let us make a few observations.

(a) Consider $\triangle BQP$ and $\triangle CQP$. Since $|\overline{BQ}| = |\overline{CQ}|$, $\angle BQP = \angle CQP = 90°$, and they share the side $\overline{PQ}$, the two triangles are congruent by the SAS (side-angle-side). In particular, this implies $|\overline{BP}| = |\overline{CP}|$.

(b) Consider $\triangle EAP$ and $\triangle FAP$. Since $\angle AEP = \angle AFP = 90°$, $\angle EAP = \angle FAP$, and they share the side $\overline{AP}$, the two triangles are congruent by the AAS (angle-angle-side). In particular, this implies $|\overline{EA}| = |\overline{FA}|$ and $|\overline{EP}| = |\overline{FP}|$.

(c) Consider $\triangle BEP$ and $\triangle CFP$. Since $\angle BEP = \angle CFP = 90°$, $|\overline{BP}| = |\overline{CP}|$ by (a), and $|\overline{EP}| = |\overline{FP}|$ by (b), the two triangles are congruent by the RHS (right angle-hypotenuse-side). In particular, this implies $|\overline{BE}| = |\overline{CF}|$.

Now, by (b) and (c), it follows that

$$|\overline{BA}| = |\overline{BE}| + |\overline{EA}| = |\overline{CF}| + |\overline{FA}| = |\overline{CA}|. \tag{1}$$

Hence, $\triangle ABC$ is isosceles. □

Clearly, Claim 1 is complete nonsense. And yet the above proof appears flawless and convincing. So where did it actually go wrong?

The problem arises from the misleading configuration in Figure 1.1. A more accurate diagram is provided in Figure 1.2.

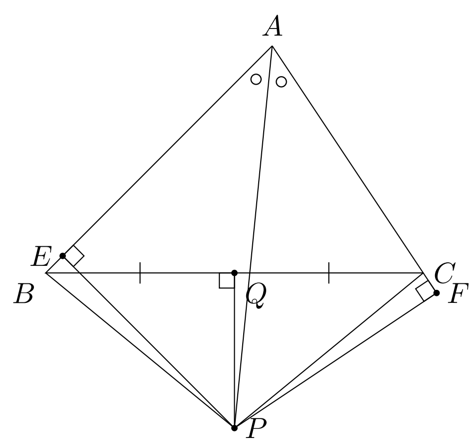

Figure 1.2

The key difference between Figures 1.1 and 1.2 is the locations of points E and F. In Figure 1.1, they are incorrectly drawn on the sides of $\triangle ABC$. But instead, point E should lie on $\overline{BA}$, and point F should locate outside $\overline{CA}$, as correctly shown in Figure 1.2.

One can now see that the arguments in (a), (b), and (c) remain valid. In particular, it is still true that

$$|\overline{BE}| = |\overline{CF}| \quad \text{and} \quad |\overline{EA}| = |\overline{FA}|.$$

However, because of the actual locations of points E and F, we can no longer conclude that $|\overline{BA}| = |\overline{CA}|$.

2 Everyone Has the Same Birthday

Claim 2. Everyone has the same birthday.

Proof. We proceed by induction on the number of people n. When there is only one person, the statement clearly holds. Assume now the statement is true for any group of k people, we want to show that it is also true for $k + 1$ people, i.e., any $k + 1$ people share the same birthday.

We label these $k + 1$ people as $1, 2, 3, \ldots, k, k + 1$. By the induction hypothesis, any k of them share the same birthday.

- Consider people labeled 1 to k. There are k individuals in total, so they share the same birthday.
- Now consider people labeled 2 to $k + 1$. Again, there are k individuals in total, so they also share the same birthday.

Since the first k people share a birthday, and so do the last k people, the overlapping individuals (2 through k) guarantee that these $k+1$ people share the same birthday. Therefore, by mathematical induction, we have proven that any group of people must all share the same birthday. □

Wait a second! What do you mean by "any group of people all share the same birthday"? That's clearly absurd! Something must be wrong—was the entire proof just nonsense?

Exactly! It was complete nonsense. Although the proof appears flawless and is easy to understand, it contains a fatal mistake. In the induction step, we assumed that any group of k people all share the same birthday, and used this to conclude that any $k + 1$ people must as well, by overlapping the first k people (1 through k) and the last k people (2 through $k+1$). However, we overlooked a crucial detail: this overlap doesn't occur when moving from $k = 1$ to $k = 2$. In that case, the two groups—person 1 and person 2—have no common individuals. So we can't conclude that they share the same birthday, and the whole argument collapses.